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A Multifractal Analysis of Equilibrium Measures for Conformal Expanding Maps and Moran-like Geometric Constructions

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In this paper we establish the complete multifractal formalism for equilibrium measures for Hölder continuous conformal expanding maps and *expanding* Markov Moran-like geometric constructions. Examples include Markov maps of an interval, beta transformations of an interval, rational maps with hyperbolic Julia sets, and conformal toral endomorphisms. We also construct a Hölder continuous homeomorphism of a compact metric space with an ergodic invariant measure of positive entropy for which the dimension spectrum is not convex, and hence the multifractal formalism fails.

KEY WORDS: Hausdorff dimension; pointwise dimension; multifractal analysis; dimension spectrum; HP spectrum; expanding map; Markov partition.

1. INTRODUCTION

Invariant sets of most dynamical systems in general are *not* self-similar in the strict sense. However, part of these sets can sometimes be decomposed into (perhaps uncountably many) subsets each supporting a Borel probability measure possessing a type of scaling symmetry. This means that the measure admits a group of scale symmetries which reproduces copies of the set (or its significant part of full measure) on arbitrarily small scales (up to a given precision which decays to zero with scale). Sets that admit such structure are called multifractals. The Hausdorff dimension of each subset can be used to characterize this structure. The detailed analysis of the

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multifractal structure of a set invariant for a chaotic dynamical system allows one to obtain a more refined description of the chaotic behavior than the description based upon purely stochastic characteristics.

This paper describes the multifractal analysis of measures invariant under dynamical systems. The concept of a multifractal analysis was suggested in the seminal paper by Halsey *et al.*,[11] who where they were attempting to understand the scaling behavior of *physical measures* on strange attractors, diffusion-limited aggregates, etc. They were also searching for quantities to *distinguish* between attractors which have the same *fractal* dimension. The multifractal analysis of measures has became a popular interdisciplinary subject of study—a search of several electronic databases showed that there are now hundreds of related papers in the physical and mathematical literature.

The first rigorous multifractal analysis of dynamical systems was carried out in ref. 5 for a special class of measures invariant under some one-dimensional Markov maps. Lopes[15] studied the measure of maximal entropy for a hyperbolic Julia set. Recently, Simpelaere[37] effected a complete multifractal analysis for equilibrium measures of Axiom A surface diffeomorphisms.

Our definition of multifractal analysis is faithful to the *definitions* in ref. 11 and other articles in the physical literature, and our work places these notions onto a solid mathematical foundation. The two major components of the multifractal analysis are the Hentschel–Procaccia (HP) spectrum for dimensions and the $f(\alpha)$ spectrum for dimensions (see descriptions below). The multifractal analysis unifies these two spectra via the Legendre transform.

There are many papers on multifractal analysis which treat only one of these two components. In a number of papers the $f(\alpha)$ spectrum for dimensions is studied not with respect to the *natural* metric, but only with respect to the *symbolic metric* (i.e., the standard metric on the symbolic space associated with the dynamical system), which is just an intermediary object and not physically meaningful. In some cases the two dimension spectra coincide, but this is a highly nontrivial result (see Theorem 2). In addition, most authors restrict their analysis to Bernoulli measures or self-similar measures and do not include measures of *actual* physical interest, like the Bowen–Ruelle–Sinai measures (BRS) on hyperbolic attractors and repellers (or general Gibbs measures). Thus, very few papers in the mathematics literature on multifractal analysis are true to the original spirit of the subject and help put it on a solid mathematical foundation.

During the last few years, there has been a burst of activity in studying the multifractal analysis for measures supported on the limit sets of geometric constructions in $\mathbb{R}^n$. Various authors have obtained some particular results for restricted classes of measures: they mostly studied the $f(\alpha)$ spectrum with respect to a symbolic metric and mostly considered Bernoulli measures. Moreover, most authors considered only similarity constructions (see extended list of references in ref. 17).

Let us say a few words for motivating the multifractal analysis. Let $g: M \to M$ be a diffeomorphism of a smooth Riemannian manifold M and let $\Lambda \subset M$ be a compact hyperbolic attractor for g. For simplicity, assume that g is topologically mixing on Λ. Bowen[1] showed that the evolution of the Lebesgue measure in a basin of Λ converges to the BRS measure. From the physical point of view, this is the natural measure on the attractor since it describes the orbit distribution of points in the basin which are typical with respect to the Lebesgue measure. This distribution is not uniform, and, as computer pictures show, there exist spots of high and low density of visits sometimes called *hot* and *cold* spots (see Fig. 1).

This phenomenon also has been observed for a more general class of attractors (hyperbolic attractors with singularities), which includes the Lorenz attractor, the Lozi attractor, etc. Attempts to analyze this measure in computer simulations are based on partitioning the basin into a very fine grid and estimating the measure of each box by the frequency with which a typical orbit visits it. This leads to an enormous amount of data.

An approach to encoding all this data was suggested in ref. 11, which who utilized the *Rényi spectrum for dimensions*, defined as follows. Cover the attractor by a *grid* of mesh size r, i.e., a partition of the repeller such that each partition element contains a ball of radius $1/2r$ and is contained

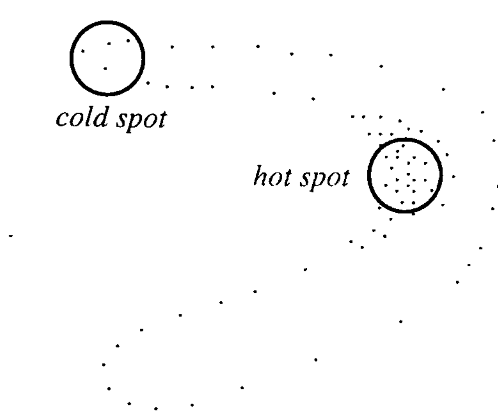

Fig. 1. Hot and cold spots on an attractor.

in a ball of radius r, where both balls are centered at the same point. Given a family of grids parametrized by mesh size r, define

$$R_\nu(q) = \frac{1}{1-q} \lim_{r \to 0} \frac{\log \sum_{i=1}^{N(r)} \nu(C_r^i)^q}{\log r}$$

provided the limit exists,[38, 39] where ν is a Borel probability measure and $N(r)$ is the number of partition elements C_r^i of the grid with $\nu(C_r^i) > 0$. A priori, the limit may depend on the family of grids. We will show that for a large class of measures (diametrically regular measures), the limit is independent of the family of grids. The result is also true if the number 1/2 in the definition of grid is replaced by any positive number.

Another approach involves the study of correlations of the distributions of q-tuples along a typical orbit for $q = 2, 3,...$. More precisely, let $g: X \to X$ be a map on a metric space (X, ρ) preserving a Borel probability measure ν. We set

$$C(x, q, r, n) = \frac{1}{n^q} \operatorname{card}\{(i_1 \cdots i_q): \rho(g^{i_j}x, g^{i_k}x) \leqslant r \text{ for all } 0 \leqslant i_j \leqslant i_k < n\}$$

We define the *correlation dimension* of order q by

$$C_q(x) = \frac{1}{1-q} \lim_{r \to 0} \lim_{n \to \infty} \frac{\log C(x, q, r, n)}{\log r}$$

provided the limits exist. If ν is ergodic, it was shown in ref. 24 (see also ref. 26) that for ν almost every x

$$\lim_{n \to \infty} C(x, q, r, n) = \int_X \nu(B(y, r))^{q-1} \, d\nu(y)$$

where $B(y, r)$ denotes the ball of radius r centered at the point y. Thus, for $q = 2, 3,...$

$$C_q(x) = \frac{1}{1-q} \lim_{r \to 0} \frac{\log \int_X \nu(B(y, r))^{q-1} \, d\nu(y)}{\log r}$$

provided the limit exists. In general, one does not expect this limit to exist. In ref. 26 the authors constructed an example of a continuous map on an interval that preserves a measure absolutely continuous with respect to the Lebesgue measure, for which the above limit does not exist for almost every x in a large interval in q. Combining this with results in ref. 12, one can construct a diffeomorphism of the two-torus preserving an ergodic measure that is absolutely continuous with respect to the Lebesgue measure, having positive topological entropy, and for which the above limit does not exist for almost every x in a large interval in q. In this paper, we show that this limit exists for a broad class of measures including equilibrium measures for conformal repellers. This unifies and extends almost all cases in the literature.

The natural extension of the correlation dimension of order $q = 2, 3,\ldots$ to all real values $q > 1$ was introduced by Hentschel and Procaccia.[10] Let ν be a Borel probability measure on a metric space (X, ρ). For $q > 1$ we define the *HP-spectrum for dimensions* by

$$\mathrm{HP}_\nu(q) = \frac{1}{1-q} \lim_{r \to 0} \frac{\log \int_X \nu(B(y, r))^{q-1}\, d\nu(y)}{\log r} \tag{HP}$$

provided the limit exists. In some cases the HP spectrum can be defined formally for all $q \neq 1$ (see Remark 5 after the statement of Theorem 3). The Rényi spectrum was a precursor to the HP spectrum where one replaces the coverings by partitions.

We work with a class of measures which incorporate the metric structure of the underlying metric space. Namely, a measure ν is *diametrically regular* or a *Federer measure*[9] if for each $A > 1$ there exists $K > 0$ such that for any sufficiently small $r > 0$ and every x we have

$$\nu(B(x, Ar)) \leqslant K\nu(B(x, r)) \tag{DR}$$

In the harmonic analysis literature such a measure is sometimes called a doubling measure. To show that a measure ν is diametrically regular, it is clearly enough to establish (DR) for a single value of $A > 1$.

We show in Theorem A2 that Gibbs measures concentrated on repellers for expanding maps are diametrically regular. This fact plays a crucial role in our multifractal analysis.

In ref. 25, Pesin showed that if ν is diametrically regular, then for any $q > 1$

$$\mathrm{HP}_\nu(q) = \frac{1}{1-q} \lim_{r \to 0} \frac{\log \inf_{\mathfrak{B}_r} \sum_{B \in \mathfrak{B}_r} \nu(B)^q}{\log r}$$

where the infinum is taken over all covers $\mathfrak{B}_r$ of X by balls B of radius r, *provided the limit exists.* We will use this definition of HP spectrum for dimensions in our proofs.

Pesin[25] showed that the Rényi spectrum coincides with the HP spectrum. In general, even *good* measures may not be diametrically regular. One can construct a smooth ergodic measure for a diffeomorphism of a compact manifold which is not diametrically regular.[25]

In this paper, we introduce and systematically study Hölder continuous conformal expanding maps on compact metric spaces. One of our main results is that any Gibbs measure corresponding to a Hölder continuous function is diametrically regular (Theorem A2). Our main tool is a construction of a Markov partition for continuous expanding maps. This construction is geometrically natural and simpler than other constructions for smooth expanding maps that we are aware of. This construction is specially adapted to a given point (or any finite collection of points) such that the partition element containing this point also contains a "large" ball centered at the point. The same approach can be used to construct special Markov partitions for Axiom A diffeomorphisms and their continuous analogs. Hence, Gibbs measures for Axiom A diffeomorphisms are diametrically regular.

We turn to the second ingredient in our multifractal analysis and define the $f_\nu(\alpha)$ spectrum for dimensions. Given $x \in X$, we consider the *upper* and *lower pointwise dimensions of* ν at x,

$$\bar{d}_\nu(x) = \limsup_{r \to 0} \frac{\log \nu(B(x, r))}{\log r}, \qquad \text{and} \qquad \underline{d}_\nu(x) = \liminf_{r \to 0} \frac{\log \nu(B(x, r))}{\log r}$$

If $\underline{d}_\nu(x) = \bar{d}_\nu(x)$, we call the common value the *pointwise dimension* at x and denote it by $d_\nu(x)$. We call ν *exact dimensional* if

$$\bar{d}_\nu(x) = \underline{d}_\nu(x) = d_\nu(x) = d$$

for ν-almost every x, where d is a nonnegative constant. In general one does not expect the pointwise dimension of ν to exist at a typical point even for *nice* measures which are invariant under dynamical systems.[16, 27] Even when the pointwise dimension of ν does exist it is not necessarily exact dimensional.[3, 27]

Nevertheless, measures which are invariant under smooth dynamical systems with hyperbolic behavior often turn out to be exact dimensional. Eckmann and Ruelle have conjectured that *hyperbolic measures* (i.e., ergodic measures with nonzero Lyapunov exponents almost everywhere) are exact dimensional. This has been established for hyperbolic measures in the two-dimensional case in ref. 41 and for hyperbolic BRS measures and equilibrium measures for Axiom A diffeomorphisms in refs. 14 and 30.

The multifractal analysis is a description of the fine-scale geometry of the set X whose constituent components are the sets

$$K_\alpha = \{x \in X \mid d_\nu(x) = \alpha\}$$

for $\alpha \in \mathbb{R}$. The $f_\nu(\alpha)$ *spectrum for dimensions* is defined by

$$f_\nu(\alpha) = \dim_H K_\alpha$$

where $\dim_H K_\alpha$ denotes the Hausdorff dimension of the set K_α. *A priori*, one may consider the box dimension instead of the Hausdorff dimension in this definition. In Remark 2 after Theorem 3 we observe that in the cases we consider in this paper, this replacement leads to a trivial spectrum of dimensions.

We obtain a natural decomposition of the set X as

$$X = \bigcup_{-\infty < \alpha < \infty} K_\alpha \cup \{x \in X \mid d_\nu(x) \text{ does not exist}\}$$

If ν is exact dimensional, then $\nu(K_d) = 1$. It is important to emphasize that the union of the sets K_α may not be all of X. Shereshevsky[36] showed that for a class of C^2 Axiom A surface diffeomorphisms, the set $X \setminus \bigcup_\alpha K_\alpha$ is dense and has positive Hausdorff dimension for any equilibrium measure ν. His proof can be modified for conformal repellers.

For the maps we consider in this paper (Hölder continuous conformal expanding maps) there exists an open interval of values of α such that the sets K_α are dense. Thus for the maps we consider this decomposition of the space X is quite complicated from the topological point of view.

In ref. 11 the authors presented a heuristic argument showing that the HP spectrum for dimensions and the $f(\alpha)$ spectrum for dimensions form a Legendre transform pair. For this to make sense one must first establish that the two spectra are smooth and strictly convex on some interval. *A priori* this seems quite amazing since in general one expects the functions $f_\nu(\alpha)$ and $\mathrm{HP}_\nu(q)$ to be only measurable. Furthermore, it is not at all clear whether, even in the exact dimensional case, the pointwise dimension attains *any* values besides d. Once the Legendre transform relation between the two dimension spectra is established, one can compute the delicate and seemingly intractable $f_\nu(\alpha)$ spectrum through the HP spectrum, which is completely determined by the statistics of a typical trajectory. See also Theorem 6.

Recently, Simpelaere[37] effected a multifractal analysis of equilibrium measures for Axiom A surface diffeomorphisms. Simpelaere's approach can presumably be modified to work for (multidimensional) conformal repellers of (noninvertible) expanding maps. In ref. 29 we give an alternative proof of his result using the methods introduced in this paper as well as compare and contrast the two approaches. Simpelaere uses methods from large-deviation theory to obtain upper estimates of the Hausdorff dimension of the level sets K_α, while our approach is based on the equality of the symbolic pointwise dimension and pointwise dimension. Simpelaere established the Legendre transform relation only for a special type of Rényi spectrum defined with respect to a special family of grids which is adapted to the hyperbolic splitting. Our approach handles the general Rényi spectrum (which we show coincides with the HP spectrum) with the help of the diametrically regular property of equilibrium measures that we establish in Theorem A2.

In Remark 3 in Section 4 we construct a Hölder continuous homeomorphism of a compact metric space with an ergodic invariant measure of positive entropy for which the dimension spectrum is not convex, and hence the multifractal formalism fails.

In this paper we effect a thorough multifractal analysis of equilibrium measures for Hölder continuous conformal expanding maps. Examples include Markov maps of an interval, hyperbolic Julia sets, and conformal toral endomorphisms. We prove that the functions $f_\nu(\alpha)$ and $(1-q)$; $\mathrm{HP}_\nu(q)$ are analytic, strictly convex on an interval, and form a Legendre transform pair, provided the measure is not the measure of maximal entropy (see Theorem 1). In particular this implies that the set of values attained by the pointwise dimension contains an open interval (α_1, α_2). Furthermore for each $\alpha \in [\alpha_1, \alpha_2]$, the set K_α is dense.

Our results generalize and extend all known results related to the multifractal analysis of smooth conformal expanding maps. We do not use any techniques from the theory of large deviations in our analysis as do refs. 5, 15, and 37. However, after effecting the multifractal analysis, we can apply a large-deviation result of Ellis[6] to obtain an interesting *counting formula* for the dimension spectrum $f_\nu(\alpha)$ (Theorem 6).

The HP spectrum of dimensions $\mathrm{HP}_\nu(q)$ is not *a priori* defined for $q=1$. It is conjectured that in "good" cases, $\lim_{q \to 1+} \mathrm{HP}_\nu(q) = I(\nu)$, where $I(\nu)$ is the information dimension (see Remark 4 after Theorem 3 for the definition). It immediately follows from our analysis that this conjecture is true for equilibrium measures for Hölder continuous conformal expanding maps.

Another class of examples that we consider are Moran-like symbolic geometric constructions where the basic sets comply with a given symbolic dynamical system. Moran first studied a special class of these constructions and computed the Hausdorff dimension of the limit set. In refs. 27 and 28 we extended the original Moran idea to much broader classes of geometric constructions.

In this paper we undertake a complete multifractal analysis of equilibrium measures for a large class of Moran-like geometric constructions which satisfy the separation condition introduced in Section 4. This separation condition allows for large intersections of the basic sets at each level. Our analysis is based upon the dynamical properties of the map on the limit set induced by the shift map on symbolic space, and we require that this map is expanding and conformal. In general, it need not be either expanding nor conformal even when the Moran-like construction is modeled by the full shift. Whether the induced map is expanding and conformal strongly depends on the symbolic dynamics and its embedding into Euclidean space, i.e., the gaps between-the basic sets. Our multifractal analysis of these *expanding and conformal* geometric constructions is intimately related to our analysis for conformal expanding maps.

We conclude this introduction by previewing a few new ideas and results that we develop in this paper. We construct a new Markov partition for repellers (see Theorem A1 in Appendix A). This construction is geometrically natural and simpler than other constructions for smooth expanding maps that we are aware of. This construction is specially adapted to a given point (or any finite collection of points) such that the partition element containing this point also contains a "large" ball centered at the point. The same approach can be used to construct special Markov partitions for Axiom A diffeomorphisms and their continuous analogs.

Using Markov partitions, we prove that equilibrium measures on conformal repellers are diametrically regular (see Theorem A2 in Appendix A). In the harmonic analysis literature such measures are sometimes called doubling measures. These measures strongly encode the metric structure of the underlying metric space. We exploit this important property many times in our analysis.

Theorem 2 is one of the major results in the paper. This theorem allows us to compute pointwise dimensions using the symbolic model and then to conclude that the pointwise dimensions computed on the symbolic level coincide with the pointwise dimensions on the repeller. The proof of Theorem 2 uses, in particular, the fact that equilibrium measures are diametrically regular.

Finally, we show that for equilibrium measures, the HP spectrum, which is *a priori* only defined for $q > 1$, is well defined (and finite) for all $q \in \mathbb{R}$. See Remark 5 after the statement of Theorem 3.

For physical motivations for the multifractal analysis, along with a complete multifractal analysis of equilibrium measures on two-dimensional hyperbolic sets based on the methods developed in this paper, we refer the reader to ref. 29. Weiss[40] gives an interesting application of the multifractal analysis to the spectrum of Lyapunov exponents. Pesin[25] provides a comprehensive and systematic treatment of dimension theory in dynamical systems.

2. CONFORMAL REPELLERS

In this section we undertake a multifractal analysis for smooth expanding maps. Let M be a smooth Riemannian manifold and $g: M \to M$ a $C^{1+\alpha}$ map. Let J be a compact subset of M such that (i) $g(J) = J$, (ii) there exists $C > 0$ and $\alpha > 1$ such that $\|dg_x^n u\| \geqslant C\alpha^n \|u\|$ for all $x \in J$, $u \in T_x M$, and $n \geqslant 1$ (for some Riemannian metric on M), and that (iii) g is topologically transitive on J. In this case we say that g is a smooth expanding map on J. If, in addition, one assumes that there exists an open neighborhood V of J (a basin) such that $J = \{x \in V: g^n \in V \text{ for all } n \geqslant 0\}$, we call J a *repeller*. The results in this paper do not require this extra condition on J. We will call J a repeller even if it does not possess an open basin.

We recall some facts about expanding maps. For simplicity we assume that the map g on J is topologically mixing.[1] Bowen[1] and Ruelle[34] show that for any Hölder continuous function ξ on J there exists a unique equilibrium measure $\nu = \nu_\xi$ on J. It is well known that expanding maps have Markov partitions consisting of partition elements called *rectangles* $\{R_1, ..., R_p\}$ of (arbitrarily small) diameter δ such that:

(1) Each rectangle R is the closure of its interior $\mathring{R}$

(2) $J = \bigcup_i R_i$

(3) $\mathring{R}_i \cap \mathring{R}_j = \varnothing$ for $i \neq j$

(4) Each $g(R_i)$ is a union of rectangles R_j

(See refs. 34 and 35 and Appendix A.) A Markov partition $\mathscr{R} = \{R_1, ..., R_p\}$ generates a symbolic model of the repeller by a subshift of finite type (Σ_A^+, σ), where $A = (a_{ij})$ is the transfer matrix of the Markov partition, i.e., $a_{ij} = 1$ if $\mathring{R}_i \cap g^{-1}(\mathring{R}_j) \neq \varnothing$ and $a_{ij} = 0$ otherwise. This defines a coding map $\chi: \Sigma_A^+ \to J$ such that the following diagram commutes:

$$\begin{tikzcd} \Sigma_A^+ \arrow[r, "\sigma"] \arrow[d, "\chi"'] & \Sigma_A^+ \arrow[d, "\chi"] \\ J \arrow[r, "g"] & J \end{tikzcd}$$

The map χ is Hölder continuous and injective on the set of points whose trajectories never hit the boundary of any element of the Markov partition.

Let ξ be a Hölder continuous function on J. The pullback by χ of ξ is a Hölder continuous function φ on Σ_A^+, i.e., $\varphi = \chi_*^{-1}\xi$. Let μ_φ be the Gibbs measure corresponding to φ. Its push forward is a measure on J which is the equilibrium measure corresponding to ξ (see Appendix B). We denote it by ν_ξ.

A Markov partition $\mathscr{R} = \{R_1, ..., R_p\}$ allows one to set up a complete analogy between repellers of expanding maps and limit sets of Moran geometric constructions (see ref. 27 and Section 4). Define the *basic sets*

$$R_{i_1 \cdots i_n} = R_{i_1} \cap g^{-1}(R_{i_2}) \cap \cdots \cap g^{-n+1}(R_{i_n}) \tag{1}$$

where g^{-i} is a branch of the inverse of g^i. By the Markov property, every basic set $r_{i_1 \cdots i_n} = r_{i_1} \cap h(r_{i_n})$ for some branch h of g^{-n+1}.

We need the following well-known estimates of the Jacobian of $C^{l+\alpha}$ expanding maps.[13, 18]

Proposition 1. There ezist positive constants C_1, C_2 such that for any n-tuple $(i_l \cdots i_n)$ and any $x, y \in R_{i_1 \cdots i_n}$

$$C_1 \leqslant \frac{|\operatorname{Jac} g^n(x)|}{|\operatorname{Jac} g^n(y)|} \leqslant C_2$$

where $\operatorname{Jac} g$ denotes the Jacobian of g.

A smooth map g is called *conformal* if $dg_x = a(x) \operatorname{Isom}_x$, where Isom_x denotes an isometry of the tangent space $T_x M$. A smooth conformal map g is called an *expanding map* if $|a(x)| > 1$ for all points x. The repeller J for a conformal expanding map g is called a *conformal repeller*. Examples of conformal repellers include one-dimensional Markov maps and hyperbolic Julia sets (see below). Ruelle[35] showed that the Hausdorff dimension d of a conformal repeller J is given by *Bowen's formula* $P(-d \log |a|) = 0$, where P is the thermodynamic pressure, and that the d-Hausdorff measure is equivalent to the equilibrium measure m corresponding to $-d \log |a|$. The measure m plays a special role in the multifractal analysis and is called the *measure of maximal dimension.*

Using the basic sets, one can construct a special *Moran cover* $\mathfrak{U}_r$ of the repeller. Given and a point $\omega \in \Sigma_A^+$, let $n(\omega)$ denote the unique positive integer such that

$$\prod_{k=0}^{n(\omega)-1} |a(\chi(\sigma^k(\omega)))|^{-1} > r, \qquad \prod_{k=0}^{n(\omega)} |a(\chi(\sigma^k(\omega)))|^{-1} \leqslant r \tag{2}$$

It is easy to see that $n(\omega) \to \infty$ as $r \to 0$ uniformly in ω. Fix $\omega \in \Sigma_A^+$ and consider the cylinder set $C_{i_1 \cdots i_{n(\omega)}} \subset \Sigma_A^+$. We have $\omega \in C_{i_1 \cdots i_{n(\omega)}}$, and if $\omega' \in C_{i_1 \cdots i_{n(\omega)}}$ with $n(\omega') \geqslant n(\omega)$, then

$$C_{i_1 \cdots i_{n(\omega')}} \subset C_{i_1 \cdots i_{n(\omega)}}$$

Let $C(\omega)$ be the largest cylinder set containing ω with the property that $C(\omega) = C_{i_1 \cdots i_{n(\omega'')}}$ for some $\omega'' \in C(\omega)$ and $C_{i_1 \cdots i_{n(\omega')}} \subset C(\omega)$ for any $\omega' \in C(\omega)$. The sets $C(\omega)$ corresponding to different $\omega \in \Sigma_A^+$ either coincide or are disjoint. We denote these sets by C_r^j, $j = 1,..., N_r$. There exist points $\omega_j \in \Sigma_A^+$ such that $C_r^j = C_{i_1 \cdots i_{n(\omega_j)}}$. These sets form a disjoint cover of Σ_A^+ which we denote by $\mathfrak{A}_r$. The sets $R_r^j = \chi(C_r^j)$, $j = 1,..., N_r$, may overlap along their boundaries. They comprise a cover of J (which we will denote by the same symbol $\mathfrak{A}_R$ if it does not cause any confusion). We have that $R_r^j = R_{i_1 \cdots i_{n(x_j)}}$ for some $x_j \in J$.

Let $Q \subset \Sigma_A^+$ be a subset. One can repeat the above arguments to construct a special Moran cover of the set $\chi(Q) \subset J$. It consists of cylinder sets C_r^j, $j = 1,..., N_r$, for which there exist points $\omega_j \in Q$ such that $C_r^j = c_{i_1 \cdots i_{n(\omega_j)}}$ and the intersection $C_r^j \cap C_r^{(i)} \cap Q$ is empty if $j \neq i$. We denote this special cover by $\mathfrak{A}_{r, Q}$.

The Moran cover has the following crucial property. Given a point $x \in J$ and a positive number r, the number of basic sets R_r^j in the Moran cover $\mathfrak{A}_r$ that have nonempty intersection with the ball $B(x, r)$ is bounded from above by a number M, which is independent of x and r. We call this number the *Moran multiplicity factor*.[27]

In order to verify this property of the Moran cover let $r_0 = \max\{\text{diam } R_i : i = 1,..., p\}$. Since the sets R_i are the closure of their interiors, there exists a number $0 < r_l < r_0$ such that each R_i contains a ball of radius r_l. The following proposition shows that each basic set R_r^j in the Moran cover contains a ball of radius Cr, where $C > 0$ is a constant independent of r and j. This implies the desired property of the Moran cover.

Proposition 2. There exist positive constants D_1 and D_2 such that for every $x \in J$

$$B\left(x, D_1 \prod_{k=0}^{n-1} |a(g^k(x))|^{-1}\right)$$
$$\subset R_{i_1 \cdots i_n}(x) \subset B\left(x, D_2 \prod_{k=0}^{n-1} |a(g^k(x))|^{-1}\right)$$

where $R_{i_1 \cdots i_n}(x)$ is a basic set containing x [see (1)] and $B(x, a)$ is the ball of radius a centered at the point x.

Proof. Since g is conformal and expanding on J, we have

$$\|dg_x^n\| = \prod_{k=0}^{n-1} |a(g^k(x))| = |\text{Jac } g^n(x)|$$

This fact and Proposition 1 imply

$$
\begin{aligned}
&\operatorname{diam} R_{i_1 \dots i_n}(x) \\
&\quad \leqslant \operatorname{diam} R_{i_n} \times \max_{y \in R_{i_n}} \|dg_y^{-n}\| \\
&\quad = \operatorname{diam} R_{i_n} \times \max_{y \in R_{i_n}} |\operatorname{Jac} g^{-n}(y)| \\
&\quad = \operatorname{diam} R_{i_n} \left(\frac{\max_{y \in R_{i_n}} |\operatorname{Jac} g^{-n}(y)|}{|\operatorname{Jac} g^{-n}(g^n(x))|} \right) |\operatorname{Jac} g^{-n}(g^n(x))| \\
&\quad \leqslant C_1 \prod_{k=0}^{n-1} |a(g^k(x))|^{-1}
\end{aligned}
$$

where $C_1 > 0$ is a constant. Since each R_j is the closure of its interior, we have that for the intrinsic diameter of $R_{i_1 \dots i_n}(x)$

$$
\begin{aligned}
&\operatorname{int\,diam} R_{i_1 \dots i_n}(x) \\
&\quad \geqslant \operatorname{diam} R_{i_n} \times \min_{y \in R_{i_n}} \|dg_y^{-n}\| \\
&\quad = \operatorname{diam} R_{i_n} \times \min_{y \in R_{i_n}} |\operatorname{Jac} g^{-n}(y)| \\
&\quad = \operatorname{diam} R_{i_n} \left(\frac{\min_{y \in R_{i_n}} |\operatorname{Jac} g^{-n}(y)|}{|\operatorname{Jac} g^{-n}(g^n(x))|} \right) |\operatorname{Jac} g^{-n}(g^n(x))| \\
&\quad \geqslant C_2 \prod_{k=0}^{n-1} |a(g^k(x))|^{-1}
\end{aligned}
$$

where $C_2 > 0$ is a constant. This completes the proof of Proposition 2. ∎

Let ξ be a Hölder continuous function on J and $\nu = \nu_\xi$ the corresponding equilibrium measure for g. Denote by φ the pull back of ξ under the coding map χ and by $\mu = \mu_\varphi$ the Gibbs measure corresponding to ξ. We have that $\nu = \chi_* \mu$.

Let ψ be the function such that $\log \psi = \varphi - P(\varphi)$. Clearly ψ is a Hölder continuous function on Σ_A^+ such that $P(\log \psi) = 0$ and μ is the Gibbs measure for $\log \psi$.

Define the one-parameter family of functions ϕ_q, $q \in (-\infty, \infty)$ on Σ_A^+ by $\phi_q(\omega) = -T(q) \log |a(\chi(\omega))| + q \log \psi(\omega)$, where $T(q)$ is chosen such that $P(\phi_q) = 0$ [one can show that $T(q)$ exists for every $q \in \mathbb{R}$; see Lemma 4 in the proof of Theorem 1 below]. It is obvious that the functions ϕ_q are Hölder continuous.

We now state our main theorem for $C^{l+\alpha}$ conformal expanding maps.

Theorem 1. The following conditions hold.

(1) The pointwise dimension $d_\nu(x)$ exists for ν-almost every $x \in J$ and

$$d_\nu(x) = \left[\int_{\Sigma_A^+} \log \psi(\omega)\, d\mu(\omega) \right] \Big/ \left[-\int_{\Sigma_A^+} \log |a(\chi(\omega))|\, d\mu(\omega) \right]$$

(2) The function $T(q)$ is real analytic for all $q \in \mathbb{R}$, $T(0) = \dim_H J$, $T(1) = 0$, $T'(q) \leqslant 0$, and $T''(q) \geqslant 0$ (see Fig. 2A).

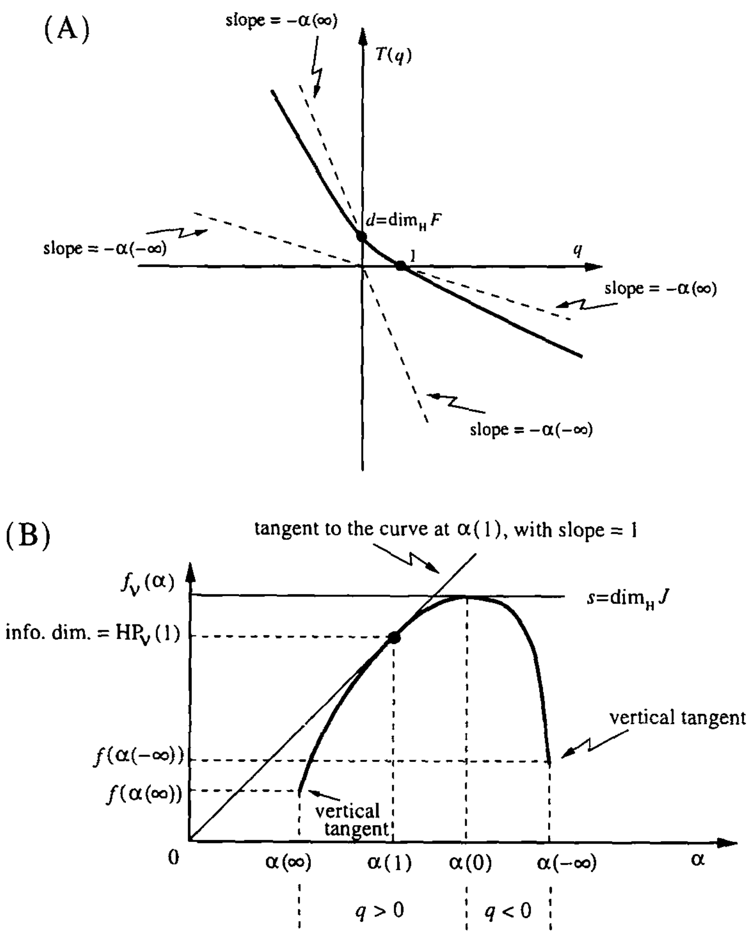

Fig. 2. (A) $T(q)$, (B) $f_\nu(\alpha)$.

(3) The function $\alpha(q) = -T'(q)$ attains values in the interval $[\alpha_1, \alpha_2]$, where $0 \leqslant \alpha_1 \leqslant \alpha_2 < \infty$. The function $f_\nu(\alpha(q)) = T(q) + q\alpha(q)$ (see Fig. 2B).

(4) If $\nu \neq m$, then the functions $f_\nu(\alpha)$ and $T(q)$ are strictly convex and form a Legendre transform pair (see Appendix C).

(5) The ν-measure of any open ball centered at points in J is positive and for any $q \in \mathbb{R}$ we have

$$T(q) = -\lim_{r \to 0} \frac{\log \inf_{\mathscr{G}_r} \sum_{B \in \mathscr{G}_r} \nu(B)^q}{\log r}$$

where the infimum is taken over all finite covers $\mathscr{G}_r$ of J by open balls B of radius r. For any $q > 1$ (actually for any $q \neq 1$; see Remark 5 after the statement of Theorem 3) we have

$$\frac{T(q)}{1-q} = HP_\nu(q) = R_\nu(q)$$

where $R_\nu(q)$ denotes the Rényi spectrum.

Proof. Fix any $q \in \mathbb{R}$. Let μ_q denote the Gibbs measure corresponding to φ_q and let ν_q denote the push forward of μ_q to J. Clearly, $T(0) = \dim_H J = d$.

To prove Statement 1, we need the following lemma.

Lemma 1. There exist constants $C_1 > 0$ and $C_2 > 0$ such that for all basic sets $R_{i_1 \ldots i_n}$,

$$C_1 \leqslant \frac{\nu_q(R_{i_1 \ldots i_n})}{m(R_{i_1 \ldots i_n})^{T(q)/d} \, \nu(R_{i_1 \ldots i_n})^q} \leqslant C_2 \tag{3}$$

Proof. Since the measures μ and μ_q are Gibbs measures corresponding to the Hölder continuous functions $\log \psi(\omega)$ and $-T(q) \log |a(\chi(\omega))| + q \log \psi(\omega)$, respectively, and m is the equilibrium measure corresponding to function $-d \log(|a(x)|)$, it follows from the definition of Gibbs measure [see (B1) in Appendix B] that the ratios

$$\frac{\nu(R_{i_1 \ldots i_n}(x)}{\prod_{k=0}^{n-1} \psi(g^k(x))}$$

$$\frac{\nu_q(R_{i_1 \ldots i_n}(x))}{\prod_{k=0}^{n-1} |a((g^k(x)))|^{-T(q)} \psi(g^k(x))^q}$$

$$\frac{m(R_{i_1 \ldots i_n}(x))}{\prod_{k=0}^{n-1} |a(g^k(x))|^{-d}}$$

are bounded from below and above by constants independent of n. The lemma easily follows. ▌

Given $0 < r < 1$, consider the Moran cover $\mathfrak{A}_r$ of the repeller J by basic sets $R_r^j = R_{i_1 \cdots i_n(x_j)}$ with radii approximately equal to r. Let $N(x, r)$ denote the number of sets R_r^j that have a non-empty intersection with a given ball $B(x, r)$ centered at x of radius r. We have that $N(x, r) \leqslant M$, uniformly in x and r, where M is the Moran multiplicity factor.

Since the measure m is an equilibrium measure and $P(-d \log |a(x)|) = 0$, there exist positive constants C_1 and C_2 such that

$$C_1 \leqslant \frac{m(R_{i_1 \cdots i_n}(x))}{\prod_{k=0}^{n-1} |a(g^k x)|^{-d}} \leqslant C_2$$

(see Appendix B).

It follows from properties of the Moran cover [see (2)] that there exist positive numbers C_5 and C_6 such that for every $R_r^j \in \mathfrak{A}_r$

$$C_5 r^d \leqslant m(R_r^j) \leqslant C_6 r^d \tag{4}$$

Since $\mathfrak{A}_r$ is a disjoint cover of J, we have

$$\sum_{R_r^j \in \mathfrak{A}_r} v_q(R_r^j) = 1$$

Summing (4) over the cover $\mathfrak{A}_r$, we obtain that there exist positive constants C_7 and C_8 such that

$$C_7 \leqslant r^{T(q)} \sum_{R_r^j \in \mathfrak{A}_r} v(R_r^j)^q \leqslant C_8$$

Taking logs and dividing by $\log r$ yields

$$-\lim_{r \to 0} \frac{\log \sum_{R_r^j \in \mathfrak{A}_r} v(R_r^j)^q}{\log r} = T(q) \tag{5}$$

We now prove Statement 1 of the theorem. Given a number $\alpha \geqslant 0$, let

$$\hat{K}_\alpha = \left\{ \omega \in \Sigma_A^+ \mid \lim_{n \to \infty} \frac{\sum_{k=0}^{n-1} \log \psi(\sigma^k(\omega))}{\sum_{k=0}^{n-1} \log |a(\chi(\sigma^k(\omega)))|^{-1}} = \alpha \right\} \tag{6}$$

Define the spectrum

$$\hat{f}_v(\alpha) = \dim_H \hat{K}_\alpha \tag{7}$$

Given $q \in \mathbb{R}$, set

$$\alpha(q) = \frac{\int_{\Sigma_A^+} \log(\psi(\omega))\, d\mu^q}{\int_{\Sigma_A^+} \log |a(\chi(\omega))|^{-1}\, d\mu_q}$$

We will show that this definition of $\alpha(q)$ coincides with Statement 3 in Theorem 1 (see Lemma 4).

The following lemma will help us to compute the Hausdorff dimension of the set $K_{\alpha(q)}$.

Lemma 2. For every $q \in \mathbb{R}$, we have:

(1) $\nu_q(\chi(\hat{K}_{\alpha(q)}) = 1$.

(2) $d_{\nu_q}(x) = T(q) + q\alpha(q)$ for ν_q-almost all $x \in \chi(\hat{K}_{\alpha(q)})$ and $\bar{d}_{\nu_q}(x) \leqslant T(q) + q\alpha(q)$ for all $x \in \chi(\hat{K}_{\alpha(q)})$.

(3) $\dim_H \chi(\hat{K}_{\alpha(q)}) = T(q) + q\alpha(q)$.

Proof. Consider the functions $\omega \mapsto \log \psi(\omega)$ and $\omega \mapsto \log |a(\chi(\omega))|$, where $\omega = (i_1 i_2 \ldots) \in \Sigma_A^+$. Since μ_q is ergodic, the Birkhoff ergodic theorem yields that

$$\lim_{n \to \infty} \frac{\log \prod_{k=1}^{n} \psi(\sigma^k(\omega))}{\log \prod_{k=1}^{n} |a(\chi(\sigma^k(\omega)))|^{-1}} = \alpha(q)$$

for μ_q-almost every $\omega \in \Sigma_A^+$. This implies the first statement.

It follows that for any $\varepsilon > 0$ and every $\omega \in \hat{K}_{\alpha(q)}$ there exists $N(\omega)$ such that for any $n > N(\omega)$

$$\alpha(q) - \varepsilon \leqslant \frac{\log \prod_{k=0}^{n-1} \psi(\sigma^k(\omega))}{\log \prod_{k=0}^{n-1} |a(\chi(\sigma^k(\omega)))|^{-1}} \leqslant \alpha(q) + \varepsilon \tag{8}$$

Given $l > 0$, denote $Q_l = \{\omega \in \hat{K}_{\alpha(q)} : N(\omega) \leqslant l\}$. It is easy to see that $Q_l \subset Q_{l+1}$ and $\hat{K}_{\alpha(q)} = \bigcup_{l=1}^{\infty} Q_l$. Thus, there exists $l_0 > 0$ such that $\mu_q(Q_l) > 0$ if $l \geqslant l_0$. Choose $l \geqslant l_0$. Given $0 < r < 1$, consider the Moran cover $\mathfrak{A}_{r, Q_l}$ of the set Q_l. It consists of cylinder sets $C^j_{r,l}$, $j = 1, \ldots, N_{r,l}$, for which there exist points $\omega_j \in Q_l$ such that $C^j_{r,l} = C_{i_1 \ldots i_{n(\omega_j)}}$. If r is sufficiently small, we have $n(\omega_j) \geqslant l$ for all j.

Since μ_q is a Gibbs measure, we obtain that for every $\omega = (i_l i_2 \ldots) \in \Sigma_A^+$ and $n > 0$

$$C_9 \leqslant \frac{\mu_q(C_{i_1 \ldots i_n})}{\prod_{k=0}^{n-1} |a(\chi(\sigma^k(\omega)))|^{-T(q)}\, \psi(\sigma^k(\omega))^q} \leqslant C_{10} \tag{9}$$

It follows from (8), (9), and the properties of the Moran cover that for all $n \geqslant l$ and any $x \in \chi(Q_l)$

$$\begin{aligned}
&\nu_q(B(x, r) \cap \chi(Q_l)) \\
&\quad \leqslant \sum_{j=1}^{M} \mu_q(C_{r,l}^{j}) \\
&\quad \leqslant C_{10} \sum_{j=1}^{M} \prod_{k=0}^{n(\omega_j)-1} |a(\chi(\sigma^k(\omega)))|^{-T(q)} \psi(\sigma^k(\omega))^q \\
&\quad \leqslant C_{10} \sum_{j=1}^{M} \prod_{k=0}^{n(\omega_j)-1} |a(\chi(\sigma^k(\omega)))|^{-T(q)-q(\alpha(q)-\varepsilon)} \\
&\quad \leqslant C_{11} r^{T(q)+q(\alpha(q)-\varepsilon)}
\end{aligned}$$

where $C_{11} > 0$ is a constant and M is the Moran multiplicity factor. Since $\nu_q(\chi(Q_l)) > 0$, for ν_q-almost every $x \in \chi(Q_l)$ there exists a number $r_0 = r_0(x)$ such that for every $0 < r \leqslant r_0$ we have

$$\nu_q(B(x, r)) \leqslant 2\nu_q(B(x, r) \cap \chi(Q_l))$$

This implies that for any $l > l_0$ and almost every $x \in \chi(Q_l)$

$$\begin{aligned}
\underline{d}_{\nu_q}(x) &= \varliminf_{r \to 0} \frac{\log \nu_q(B(x, r))}{\log r} \\
&\geqslant \varliminf_{r \to 0} \frac{\log \nu_q(B(x, r) \cap \chi(Q_l))}{\log r} \\
&\geqslant T(q) + q(\alpha(q) - \varepsilon)
\end{aligned}$$

Since sets Q_l are nested and exhaust the set Q, we obtain that $\underline{d}_{\nu_q}(x) \geqslant T(q) + q(\alpha(q) - \varepsilon)$ for ν_q-almost every $x \in \chi(\hat{K}_{\alpha(q)})$. Since ε is arbitrary, this implies that $\underline{d}_{\nu_q}(x) \geqslant T(q) + q\alpha(q)$ for ν_q-almost every $x \in (\hat{K}_{\alpha(q)})$. In particular, $\dim_H \chi(\hat{K}_{\alpha(q)}) \geqslant T(q) + q\alpha(q)$.

Fix $0 < r < 1$. For each $\omega = (i_1 i_2 \cdots) \in Q_l$ choose $n(\omega)$ according to (2). It follows that $R_{i_1 \cdots i_{n(\omega)}} \subset B(x, 2D_2 r)$, where $x = \chi(\omega)$. By virtue of (8) and (9), for all $\omega \in Q_l$

$$\begin{aligned}
&\nu_q(B(x, 2D_2 r)) \\
&\quad \geqslant \nu_q(R_{i_1 \cdots i_{n(\omega)}}) \\
&\quad \geqslant C_9 \prod_{k=0}^{n(\omega)-1} |a(\chi(\sigma^k(\omega)))|^{-T(q)} \psi(\sigma^k(\omega))^q
\end{aligned}$$

$$\geqslant C_9 \prod_{k=0}^{n(\omega)-1} |a(\chi(\sigma^k(\omega)))|^{-T(q)-q(\alpha(q)+\varepsilon)}$$

$$\geqslant C_9 r^{T(q)+q(\alpha(q)+\varepsilon)}$$

It follows that for all $x \in \chi(Q_l)$

$$\bar{d}_{\nu_q}(x) = \overline{\lim_{r \to 0}} \frac{\log \nu_q(B(x,r))}{\log r} \leqslant T(q) + q(\alpha(q) + \varepsilon)$$

Since ε is arbitrary, this implies that $\bar{d}_{\nu_q}(x) \leqslant T(q) + q\alpha(q)$ for every $x \in \hat{K}_{\alpha(q)}$. Therefore, $\underline{d}_{\nu_q}(x) = \bar{d}_{\nu_q}(x) = T(q) + q\alpha(q)$ for ν_q-almost every $x \in \hat{K}_{\alpha(q)}$. This implies the second statement of the lemma. The second statement trivially implies that $\dim_H \hat{K}_{\alpha(q)} \geqslant T(q) + q\alpha(q)$. Moreover, by a folklore theorem,[8] the fact that $\bar{d}_{\nu_q}(x) \leqslant T(q) + q\alpha(q)$ for *every* $x \in \hat{K}_{\alpha(q)}$ implies that $\dim_H \hat{K}_{\alpha(q)} \leqslant T(q) + q\alpha(q)$. Combining these two estimates we have shown that $\dim_H \hat{K}_{\alpha(q)} \leqslant T(q) + q\alpha(q)$ and this completes the proof of the lemma. ∎

The above arguments also imply that the function $\alpha(q) \geqslant 0$ for all q.

It immediately follows from (5) that $T(1) = 0$ and thus $\mu = \mu_1$. The first statement of the theorem now follows from Lemma 2 and the following fundamental theorem, which says that the pointwise dimension we compute using the symbolic model coincides with the pointwise dimension on the repeller.

Theorem 2. (1) For every $q \in \mathbb{R}$ and every $\omega \in \hat{K}_{\alpha(q)}$ we have that $d_\nu(x) = \alpha(q)$, where $x = \chi(\omega)$.

(2) For every $q \in \mathbb{R}$ and every $x \in K_{\alpha(q)}$ there exists $\omega \in \hat{K}_{\alpha(q)}$ such that $\chi(\omega) = x$.

In other words, for all q, $\chi(\hat{K}_{\alpha(q)}) = K_{\alpha(q)}$.

Proof. Since g is a smooth expanding map on J, there exist positive constants r_0, and a such that if $x, y \in J$ and $d(x, y) < r_0$, then $d(g(x), g(y)) > ad(x, y)$. One can easily derive from Proposition 1 that given $0 < r < r_0$, there exists $N(r) > 0$ and positive constants C_{12} and C_{13} such that if $0 \leqslant n \leqslant N(r)$, then for all $x \in J$

$$C_{12} r \prod_{k=0}^{n-1} |a(g^k(x))| \leqslant \operatorname{diam}(g^n(B(x,r))) \leqslant C_{13} r \prod_{k=0}^{n-1} |a(g^k(x))| \qquad (10)$$

and there exist positive constants C_{14} and C_{15} such that for all $x \in J$ and any $n \geqslant 0$

$$C_{14} r \prod_{k=0}^{n-1} |a(g^k(x))|^{-1} \leqslant \operatorname{diam}(h(B(x, r))) \leqslant C_{15} r \prod_{k=0}^{n-1} |a(g^k(x))|^{-1} \quad (11)$$

where h denotes any branch of g^{-n}.

Fix $x \in J$ and $0 < r \leqslant r_0$ and choose $N = N(x, r)$ such that

$$C_{13} r \prod_{k=0}^{N-1} |a(g^k(x))| \leqslant r_0, \qquad C_{13} r \prod_{k=0}^{N} |a(g^k(x))| > r_0 \quad (12)$$

It easily follows that $N(x, r) \leqslant N(r)$. By virtue of (10) and (11), we have that

$$\begin{aligned} \operatorname{diam}(g^N(B(x, r)) &\leqslant C_{13} r \prod_{k=0}^{N-1} |a(g^k(x))| \\ &\leqslant r_0 = \operatorname{diam}(B(g^N(x), r_0)) \end{aligned}$$

This immediately implies that $B(x, r) \subset h(B(g^N(x), r_0))$, where h is an appropriate branch of g^{-N}. It follows from (11) that

$$\begin{aligned} \operatorname{diam}(h(B(g^N(x), r_0))) &\leqslant C_{12} r_0 \prod_{k=0}^{N-1} |a(g^k(x))|^{-1} \\ &\leqslant C_{16} r = \operatorname{diam}(B(x, C_{16} r)) \end{aligned}$$

where $C_{16} > 0$ is a constant. This implies

$$B(x, r) \subset h(B(g^N(x), r_0)) \subset B(x, C_{16} r)$$

Consider the special Markov partition $\mathscr{R}_{g^N(x)}$ for the map g constructed in Theorem 6 (see Appendix A) with diameter r_0. There exist positive constants C_{17} and C_{18} such that

$$B(g^N(x), C_{17} r_0) \subset R(g^N(x)) \subset B(g^N(x), C_{18} r_0)$$

where $R(g^N(x))$ denotes the rectangle that contains the point $g^N(x)$. This implies that

$$h(B(g^N(x)), C_{17} r_0)) \subset h(R(g^N(x))) \subset h(B(g^N(x), C_{18} r_0))$$

Since the measure ν is g-invariant, we have

$$\nu(B(g^N(x), C_{17} r_0)) \leqslant \nu(h(R(g^N(x)))) \leqslant \nu(B(g^N(x), C_{18} r_0)).$$

Using the fact that the measure ν is diametrically regular (see Theorem A2 in Appendix A), we obtain

$$\begin{aligned} \nu(B(g^N(x), C_{18} r_0)) &\leqslant C_{19} \nu(B(g^N(x), r_0)) \\ &= C_{17} \nu(h(B(g^N(x), r_0))) \\ &\leqslant C_{19} \nu(B(x, C_{14} r)) \leqslant C_{20} \nu(B(x, r)) \end{aligned}$$

where $C_{19} > 0$ and $C_{20} > 0$ are constants. Similarly, we obtain

$$\begin{aligned} \nu(B(g^N(x), C_{17} r_0)) &\geqslant C_{21} \nu(B(g^N(x), r_0)) \\ &= C_{19} \nu(h(B(g^N(x), r_0))) \\ &\geqslant C_{22} \nu(B(x, r)) \end{aligned}$$

where $C_{21} > 0$ and $C_{22} > 0$ are constants. Thus,

$$C_{22} \nu(B(x, r)) \leqslant \nu(h(R(g^N(x)))) \leqslant C_{20} \nu(B(x, r))$$

Since ν is a Gibbs measure and $h(R(g^N(x)))$ is a basic set, its measure is given via symbolic dynamics. We obtain

$$\begin{aligned} d_\nu(x) &= \lim_{r \to 0} \frac{\log \nu(B(x, r))}{\log r} \\ &= \lim_{N(r) \to \infty} \frac{\log \prod_{k=0}^{N(r)-1} \psi(\sigma^k(\omega))}{\log \prod_{k=0}^{N(r)-1} |a(\chi(\sigma^k(\omega)))|^{-1}} \end{aligned}$$

Part (1) of the lemma immediately follows.

Now assume that $d_\nu(x) = \alpha(q)$. We need to show that the existence of the subsequential limit as $N(r) \to \infty$ implies the existence of the limit as $n \to \infty$. Consider the sequence $r_k = 2^{-k}$. It follows from the definition of $N(r)$ and a crude estimate of $|a(y)|$ that there exist positive constants C_{23} and C_{24} such that $N(r_{k+1}) - N(r_k) \leqslant C_{23} + C_{24} k$. Part (2) of the lemma immediately follows. ∎

We now prove Statements 2–4 of Theorem 1. We first note that $\dim_H K_{\alpha(q)} = T(q) + q\alpha(q)$. Since $\nu_q(K_{\alpha(q)}) = 1$, this is a consequence of Lemmas 2 and 3 and the following general result.

Proposition 3. Let (X, ρ) be a complete separable metric space of finite topological dimension with metric ρ, and let μ be a Borel probability measure. If $Z_\beta = \{x \in X \mid \underline{d}_\mu(x) = \bar{d}_\mu(x) = \beta\}$ and $\mu(Z_\beta) > 0$, then $\dim_H Z_\beta = \beta$.

Proof. It follows from Young's result[41] that $\dim_H \mu = \beta$. This immediately implies that $\dim_H Z_\beta \geqslant \dim_H \mu \geqslant \beta$.

Fix $\gamma > 0$. It follows from the definition of pointwise dimension that for any $x \in Z_\beta$ there exists $\varepsilon(x) > 0$ such that

$$\mu(B(x, \varepsilon)) \geqslant \varepsilon^{\beta + \gamma}$$

for any $\varepsilon \leqslant \varepsilon(x)$. Define $Z_{\beta, r} = \{x \in Z_\beta \mid \varepsilon(x) \geqslant r^{-1}\}$. Clearly, $Z_\beta = \bigcup_{r=1}^{\infty} Z_{\beta, r}$. It follows from the Frostman theorem[8] that $\dim_H Z_\beta \leqslant \beta + \gamma$. Since this inequality holds for all $\gamma > 0$, the desired result follows. ∎

We need the following lemmas.

Lemma 3. The function $T(q)$ is real analytic for all $q \in \mathbb{R}$.

Proof. Consider the function $c: \mathbb{R}^2 \to C^\alpha(\Sigma_A^+, \mathbb{R})$ defined by $c(q, r) = q \log \psi + r \log |a \circ \chi|$. This function is clearly real analytic and $\phi_q = c(q, -T(q))$. Since the pressure P is real analytic,[34] the desired lemma follows immediately from the Implicit Function Theorem once we verify the nondegeneracy hypothesis. For that we use Ruelle's formula for the derivative of pressure (see Appendix B). The nondegeneracy condition is

$$\left.\frac{\partial P(c(q, r))}{\partial r}\right|_{(q, -T(q))} = \int_{\Sigma_A^+} \log |a(\chi(\omega))| \, d\mu_q \neq 0 \quad ∎ \tag{13}$$

Lemma 4. For all q we have $\alpha(q) = -T'(q)$.

Proof. Recall that $\phi_q = c(q, -T(q))$, where the function c was defined in Lemma 3. Since $P(\phi_q) = 0$ for all q, we have

$$\frac{d}{dq} P(\phi_q) = \frac{\partial P(c(q, r))}{\partial q} + \left.\frac{\partial P(c(q, r))}{\partial r}\right|_{r = -T(q)} T'(q) = 0$$

Using the formula for the derivative of pressure, we obtain that

$$\begin{aligned} T'(q) &= -\frac{\partial P(c(q, r))/\partial q|_{r = -T(q)}}{\partial P(c(q, r))/\partial r|_{r = -T(q)}} \\ &= -\frac{\int_J \log(\psi(x)) \, d\nu_q}{\int_J \log |a(x)| \, d\nu_q} = -\alpha(q) \quad ∎ \end{aligned}$$

Lemma 5. The function $T(q)$ is convex. It is strictly convex if and only if $\nu \neq m$.

Proof. Using the chain rule to compute the second derivative $(\partial^2/\partial q^2)\, P(c(q, r))$ evaluated at the point $(q, r) = (q, -T(q))$ [recalling that $P(\phi_q) = 0$], we obtain that

$$T''(q) = \left[T'(q)^2 \left(\frac{\partial^2 P(c(q, r))}{\partial r^2}\right) - 2T'(q) \left(\frac{\partial^2 P(c(q, r))}{\partial q\, \partial r}\right) + \left(\frac{\partial^2 P(c(q, r))}{\partial q^2}\right)\right] \times \left(\frac{\partial P(c_{q,r})}{\partial r}\right)^{-1}$$

evaluated at $(q, r) = (q, -T(q))$.

Ruelle[34] explicitly computed the second derivative of pressure for the shift mapping on Σ_A^+ and showed that

$$\left.\frac{\partial^2 P(h + \varepsilon_1 f_1 + \varepsilon_2 f_2)}{\partial \varepsilon_1\, \partial \varepsilon_2}\right|_{\varepsilon_1 = \varepsilon_2 = 0} = Q_h(f_1, f_2)$$

where Q_h is the bilinear form on $C^\alpha(\Sigma_A^+, \mathbb{R})$ defined by

$$Q_h(f_1, f_2) = \sum_{k=0}^{\infty} \left(\int_{\Sigma_A^+} f_1 \cdot (f_2 \circ \sigma^k)\, d\mu_h - \int_{\Sigma_A^+} f_1\, d\mu_h \int_{\Sigma_A^+} f_2\, d\mu_h \right)$$

and μ_h is the Gibbs measure for the potential h. Ruelle also shows that $Q_h(f, f) \geqslant 0$ for all f and that $Q_h(f, f) > 0$ if and only if f is not cohomologous to a constant function.

Applying this second derivative formula to compute the three second partial derivatives in the expression for $T''(q)$, we obtain that

$$T''(q) = Q_q(\log \psi(\omega) - T'(q) \log |a(\chi(\omega))|, \log \psi(\omega) - T'(q) \log |a(\chi(\omega))|) \times \left[\int_J \log |a(\chi(\omega))|\, d\mu_q(\omega) \right]^{-1}$$

where Q_q is the bilinear form defined on $C^\alpha(\Sigma_A^+, \mathbb{R})$ by

$$Q_q(f_1, f_2) = \sum_{k=0}^{\infty} \left(\int_{\Sigma_A^+} f_1 \cdot (f_2 \circ \sigma^k)\, d\mu_q - \int_{\Sigma_A^+} f_1\, d\mu_q \int_{\Sigma_A^+} f_2\, d\mu_q \right)$$

It follows that $T''(q) > 0$ provided that the function $\log \psi(\omega) - T'(q) \log |a(\chi(\omega))|$ is not cohomologous to a constant function. This can be assured provided that the functions $\log \psi(\omega)$ and $C \log |a(\chi(\omega))|$ are not cohomologous for any positive constant $C > 0$. On the other hand, if there exists $C > 0$ such that the functions $\log \psi(\omega)$ and $C \log |a(\chi(\omega))|$ are cohomologous, it follows that $C = d$ and thus $\nu = m$. This implies that $T'(q) = d$ for all $q \in \mathbb{R}$ and hence $T(q) = (1 - q)\, d$ is a linear function. ∎

It follows from Lemmas 3–5 that the function $\alpha(q)$ is analytic and $\alpha'(q) = -T''(q) < 0$. Hence, the range of the function $\alpha(q)$ contains an interval. This implies Statements 2–4 of Theorem 1.

We now prove the final statement of the theorem. Given $r > 0$, consider the Moran cover $\mathfrak{A} = \{R_r^j\}$ of J. There are positive constants C_{25} and C_{26} independent of r such that for every j one can find a point $x_j \in R_r^j$ satisfying

$$B(x_j, C_{25}r) \subset R_r^j \subset B(x_j, C_{26}r) \tag{14}$$

Since the measure ν is diametrically regular, it follows from (14) that for every $q \in \mathbb{R}$

$$\sum_j \nu(B(x_j, C_{26}r))^q \leqslant C_{27} \sum_j \nu(B(x_j, C_{25}r))^q \leqslant C_{27} \sum_j \nu(R_r^j)^q \tag{15}$$

where $C_{27} > 0$ is a constant independent of j and r.

Let $\mathscr{G}_r$ be a cover of the repeller J by balls $B(y_i, r)$. For each $j \geqslant 0$, there exists $B(y_{i_j}, r) \in \mathscr{G}_r$ such that $B(y_{i_j}, r) \cap R_r^j \neq \varnothing$. Consider the new cover of J by the balls $B_j = B(y_{i_j}, 2C_{26}r)$. By (14), each basic set R_r^j is contained in at least one element of the new cover.

Define an equivalence relation on the basic sets R_r^j by saying that two basic sets are equivalent if they are both contained in the same element of the new cover. By (14), each equivalence class contains at most K elements, where K is a constant independent of r and j. For each equivalence class ξ_k determined by some ball B_k, we have,by (14), that for any $q \geqslant 0$

$$\sum_{R_r^j \in \xi_k} \nu(R_r^j)^q \leqslant K\nu(B_k)^q \tag{16}$$

Since the measure ν is diametrically regular, we obtain by (14) that for any $q < 0$

$$\sum_{R_r^j \in \xi_k} \nu(R_r^j)^q \leqslant C_{28}\,\nu(B_k)^q \tag{17}$$

where $C_{28} > 0$ is a constant. Exploiting again the fact that the measure ν is diametrically regular, we conclude using (16) and (17) that for all $q \in \mathbb{R}$,

$$\begin{aligned}\sum_{R_r^j \in \mathfrak{A}_r} \nu(R_r^j)^q &\leqslant KC_{28}^q \sum_k \nu(B_k)^q \\ &\leqslant C_{29} \sum_k \nu(B_k)^q \leqslant C_{30} \sum_{B \in \mathscr{G}_r} \nu(B)^q \end{aligned} \tag{18}$$

where $C_{29} > 0$ and $C_{30} > 0$ are constants. Statement 6 of the theorem follows immediately from (15), and (18). ∎

It follows from Statement 1 of Theorem 1 that for ν-almost every $x \in J$

$$d_\nu(x) = \frac{h_\nu(g)}{\lambda_\nu}$$

where $h\nu(g)$ is the measure-theoretic entropy of g and $\lambda_\nu > 0$ is the Lyapunov exponent of measure ν, i.e.,

$$\lambda_\nu = \lim_{n \to \infty} \frac{\log \|dg_x^n\|}{n} = \lim_{n \to \infty} \frac{\sum_{k=1}^{n-1} \log |a(g^k(x))|}{n} = \int_J \log |a(x)| \, d\nu(x)$$

In the next theorem we summarize some refined properties of the pointwise dimension of equilibrium measures.

Theorem 3. Let $\nu = \nu_\xi$ be the equilibrium measure corresponding to a Hölder continuous function ξ on a conformal repeller J. Then for every $\alpha_1 \leqslant \alpha \leqslant \alpha_2$ we have the following:

(1) The set $\chi(\hat{K}_\alpha) = K_\alpha$ (Theorem 2).

(2) There is a unique equilibrium measure ν_α on J such that $\nu_\alpha(K_\alpha) = 1$ and $d_{\nu_\alpha}(x) = \alpha$ for every point $x \in K_\alpha$.

(3) The measure $\nu_1 = \nu$ and the measure ν_0 is the measure of maximal dimension, i.e., ν_0 is the equilibrium measure for the potential $d \log |a(x)|$, where $d = \dim_H J$.

Remarks. (1) Assume that $\nu = m$. In the proof of Lemma 5 we showed that this implies that $T(q) = (1 - q)\, d$. Since the pointwise dimension of m is equal to d everywhere in J, we have that $f_\nu(d) = d$ and $f_\nu(\alpha) = 0$ for all $\alpha \neq d$.

(2) Recall that $\nu_\alpha(K_\alpha) = 1$ for each $\alpha \in [\alpha_1, \alpha_2]$, where ν_α is an equilibrium measure. Since equilibrium measures are fully supported on J and assign positive measure to all open subsets, it follows that the sets K_α are dense in J.

It is a well known property of box dimension that the box dimension of a set coincides with the box dimension of the closure of the set.[8] Since the sets K_α are dense in J for $\alpha \in [\alpha_1, \alpha_2]$, it follows that the box dimension of these sets k_α.are equal to the box dimension of the set J. This observation shows that the multifractal analysis becomes trivial if Hausdorff dimension is replaced by box dimension in the definition of the dimension spectrum.

(3) Assume that $\nu \neq m$. The function $f_\nu(\alpha)$ is defined on the interval $[\alpha_1, \alpha_2]$, where

$$\alpha_1 = -\lim_{q \to +\infty} T'(q), \qquad \alpha_2 = -\lim_{q \to -\infty} T'(q)$$

It follows that $f(\alpha) \leqslant d = \dim_H J$ for any $\alpha \geqslant 0$ and $f(\alpha(0)) = T(0) = d$. Since the function $f_\nu(\alpha)$ is strictly convex, $\alpha(0)$ is the only value where this function attains its maximum d (see Fig. 2B).

Differentiating the equality $f(\alpha(q)) = T(q) + q\alpha(q)$ with respect to q, we obtain that for every $q \in \mathbb{R}$

$$f'_\alpha(\alpha(q)) = q$$

This implies that

$$\lim_{\alpha \to \alpha_1} f'(\alpha) = +\infty, \qquad \lim_{\alpha \to \alpha_2} f'(\alpha) = -\infty$$

Furthermore, $f'_\alpha(\alpha(1)) = 1$. Since $T(1) = 0$ we have that $f(\alpha(1)) = \alpha(1)$. Since $f_\nu(\alpha)$ is strictly convex, we conclude that the equation $f_\nu(\alpha) = \alpha$ has the unique root $\alpha(1)$. Moreover, the function $f_\nu(\alpha)$ is tangent to the line of slope 1 at $\alpha(1)$ (see Fig. 2B).

(4) Theorem 1 can be generalized to Hölder continuous expanding and conformal maps. A continuous map $g: X \to X$ on a compact metric space X is said to be *expanding* if g is a local homeomorphism and there exist constants $b > a > 1$ and ${}_0 > 0$ such that

$$B(g(x), ar) \subset g(B(x, r)) \subset B(g(x), br) \tag{19}$$

for every $x \in X$ and $0 < r < r_0$.

We say that a Hölder continuous expanding map g is *conformal* if there exist a Hölder continuous function $a(x)$ with $|a(x)| > 1$ on X and positive constants C_1, C_2, and r_0 such that for any $0 < r \leqslant r_0$, any two points x, $y \in X$, and any integer $n \geqslant 0$ we have: if $\rho(g^k(x), g^k(y)) \leqslant r_0$ for all $k = 0, 1, \ldots, n$, then

$$C_1 \prod_{k=0}^{n} |a(g^k(x))|^{-1} \leqslant \rho(g^n(x), g^n(y)) \leqslant C_2 \prod_{k=0}^{n} |a(g^k(x))|^{-1} \tag{20}$$

We denote by m the Gibbs measure corresponding to the function $-d \log |a(x)|$ on X, where d is the unique root of Bowen's equation $P(-d \log |a(x)|) = 0$.

Let ν be the Gibbs measure corresponding to a Hölder continuous function ξ on X. Define $\psi = \xi - P(\xi)$. Clearly ψ is a Hölder continuous function on X such that $P_X(\log \psi) = 0$ and ν is the unique equilibrium measure for $\log \psi$.

Define the one-parameter family of functions ξ_q, $q \in (-\infty, \infty)$, on X by

$$\xi_q(x) = -T(q) \log |a(x)| + q \log \psi(x)$$

where $T(q)$ is chosen such that $P(\xi_q)=0$. One can show that for every $q \in \mathbb{R}$ there exists only one number $T(q)$ with the above property. It is obvious that functions ξ_q are Hölder continuous on X. The following statement effects the complete multifractal analysis of Gibbs measures supported on repellers for Hölder continuous conformal expanding maps. Its proof is similar to the proof of Theorem 1 and uses Theorems A1 and A2 in Appendix A.

Theorem 4. Let g be a Hölder continuous conformal expanding map of a compact set X. Then for any Hölder continuous function ξ on X we have the following:

(1) The pointwise dimension $d_\nu(x)$ exists for ν-almost every $x \in X$ and

$$d_\nu(x)=\frac{\int_F \log \psi(x)\, d\nu(x)}{-\int_X \log |a(x)|\, d\nu(x)}$$

where $\nu=\nu_\xi$ is the Gibbs measure corresponding to ξ.

(2) The function $T(q)$ is real analytic for all $q \in \mathbb{R}$, $T(0)=\dim_H F$, and $T(1)=0$, $T'(q)\leqslant 0$, and $T''(q)\geqslant 0$ (see Fig. 2A).

(3) The function $\alpha(q)=-T'(q)$ attains values in the interval $[\alpha_1, \alpha_2]$, where $0\leqslant\alpha_1\leqslant\alpha_2<\infty$. The function $f_\nu(\alpha(q))=T(q)+q\alpha(q)$. (see Fig. 2B).

(4) If ν is not the measure of maximal entropy or $\nu \neq m$, then the functions $f_\nu(\alpha)$ and $T(q)$ are strictly convex and form a Legendre transform pair (see Appendix C).

(5) The ν-measure of any open ball centered at points in X is positive and for any $q \in \mathbb{R}$ we have

$$T(q)=-\lim_{r\to 0}\frac{\log \inf_{\mathscr{G}_r} \sum_{B\in \mathscr{G}_r} \nu(B)^q}{\log r}$$

where the infimum is taken over all finite covers $\mathscr{G}_r$ of X by open balls of radius r. For $q>1$ (actually for any $q\neq 1$; see Remark 5 after the statement of Theorem 3) we have that

$$\frac{T(q)}{1-q}=HP_\nu(q)=R_\nu(q)$$

(5) For an arbitrary Borel probability measure ν on a metric space X, the HP-spectrum[10] is not *a priori* defined for $q<1$. One problem is that

the measure of some small balls may be zero. However, if all balls have positive measure (as in the case of equilibrium measures for conformal repellers), the definition of HP spectrum for all $q \neq 1$ makes formal sense although the integral may be infinite.

In our proof of Statement 5 of Theorem 1, we actually show that for all $q \neq 1$ (not just for $q > 1$ as stated), the function $T(q)/(1-q)$ coincides with this extended definition of $\mathrm{HP}_\nu(q)$. In particular this implies that $\mathrm{HP}_\nu(q)$ is finite for all $q \neq 1$.

The case $q = 1$ is treated in Remark 6.

(6) We define the notion of information dimension. Let ξ be a finite partition of the space X. Given a Borel finite measure ν on X, the *entropy of ξ with respect to ν* is defined as

$$H_\nu(\xi) \overset{\text{def}}{\equiv} -\sum \nu(C_\xi) \log \nu(C_\xi)$$

where C_ξ is an element of the partition ξ. Given a positive number ε, we set

$$H_\nu(\varepsilon) = \inf_\xi \{H_\nu(\xi) : \operatorname{diam} \xi \leqslant \varepsilon\}$$

where $\operatorname{diam} \xi = \max \operatorname{diam} C_\xi$.

We define the *information dimension* of ν by

$$I(\nu) \overset{\text{def}}{\equiv} \lim_{\varepsilon \to 0} \frac{H_\nu(\varepsilon)}{\log(1/\varepsilon)}$$

provided that the limit exists.

Young[41] showed that if $\underline{d}_\nu(x) = \bar{d}_\nu(x) = d$ for ν-almost every $x \in X$, then $I(\nu) = d$ and hence is equal to the Hausdorff dimension of ν.

Assume that the measure ν is diametrically regular. It is conjectured that in "good" cases

$$I(\nu) = \lim_{q \to 1+} R_\nu(q) = \lim_{q \to 1+} \mathrm{HP}_\nu(q)$$

Since the function $T(q)$ is differentiable, the limit

$$\lim_{q \to 1} \frac{T(q)}{1-q}$$

exists and is equal to $-T'(1)=\alpha(1)$. It follows from Statement 5 of Theorem 1 that

$$-T'(1)=\lim_{r\to 0}\frac{\log \inf_{\mathscr{G}_r}\sum_{B\in\mathscr{G}_r}\nu(B)\log\nu(B)}{\log r}$$

where the infinum is taken over all finite covers $\mathscr{G}_r$ of J by open balls of radius r. This implies that

$$f_\nu(\alpha(1))=\alpha(1)=-T'(1)=I(\nu)$$

3. EXAMPLES

Theorem 1 allows us to effect a multifractal analysis of equilibrium measures for hyperbolic rational maps, one-dimensional Markov maps, and conformal toral endomorphisms. We first consider rational maps.

Let $R\colon \hat{\mathbb{C}}\to\hat{\mathbb{C}}$ be a rational map of degree $d\geqslant 2$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere. The map R, being holomorphic, is clearly conformal. The *Julia set* J_R of R is the closure of the set of repelling periodic points of R [recall that a periodic point p of period m is repelling if $|(R^m)'(p)|>1$]. One says that R is *hyperbolic* (or that the Julia set is *hyperbolic*) if the map R is expanding on J_R, i.e., if it satisfies conditions (1)–(3) in the definition of smooth expanding map with respect to the spherical metric on $\hat{\mathbb{C}}$.[4] It is known that the map $z\to z^2+c$ is hyperbolic provided $|c|<1/4$. It is conjectured that a dense set of rational maps is hyperbolic. Since the Julia set of a hyperbolic rational map is a conformal repeller, Theorem 1 immediately implies the following statement.

Corollary 1. If ν is an equilibrium measure for a hyperbolic rational map, then Statements 1–4 of Theorem 1 hold.

We now consider one-dimensional Markov maps. Let g be a *Markov map* of the interval $I=[0,1]$. This means that there exists a finite family $I_1, I_2,\ldots, I_p\subset I$ of disjoint closed intervals such that:

(1) for every $1\leqslant j\leqslant M$, there is a subset $K=K(j)$ of indices with $g(I_j)=\bigcup_{k\in K} I_k \bmod 0$.

(2) For every $x\in\bigcup_j \mathring{I}_j$, the derivative of g exists and satisfies $|g'(x)|\geqslant\alpha$ for some fixed $\alpha>0$.

(3) There exists $\lambda>1$ and $n_0>0$ such that if $g^m(x)\in\bigcup_j\mathring{I}_j$, for all $0\leqslant m\leqslant n_0-1$, then $|(g^{n_0})'(x)|\geqslant\lambda$.

Let $J = \{x \in I \mid g^n(x) \in \bigcup_j \mathring{I}_j \text{ for all } n \in \mathbb{N}\}$. The set J is a repeller for the map g. It is conformal because the domain of g is one-dimensional. Hence, Theorem 1 immediately implies the following statement.

Corollary 2. If ν is an equilibrium measure for a Markov map, then Statements 1–4 of Theorem 1 hold.

Rand[31] carried out a partial multifractal analysis of equilibrium measures for a *cookie-cutter map*. Cookie-cutter maps are a special type of Markov map where one has only two subintervals which get mapped onto I under g. He studied the dimension spectrum only with respect to the symbolic metric.

Another example of a conformal expanding map is a conformal toral endomorphism defined by a diagonal matrix $(m, \ldots, m)$, where m is an integer and $|m| > 1$.

Corollary 3. If ν is an equilibrium measure for a conformal toral endomorphism, then Statements 1–4 of Theorem 1 hold.

4. MULTIFRACTAL ANALYSIS OF EQUILIBRIUM MEASURES ON LIMIT SETS ON MORAN-LIKE GEOMETRIC CONSTRUCTIONS

About 50 years ago, Moran[19] computed the Hausdorff dimension of geometric constructions in $\mathbb{R}^n$ given by p basic sets $\Delta_{i_1 \cdots i_n}$ satisfying the following:

(1) Each basic set is the closure of its interior.

(2) At each level the basic sets do not overlap (their interiors are disjoint).

(3) A basic set $\delta_{i_1 \cdots i_n j}$ is geometrically similar to the basic set $\Delta_{i_1 \cdots i_n}$ for every j and n.

(4) $\operatorname{diam}(\delta_{i_1 \cdots i_n j}) = \lambda_j \operatorname{diam}(\delta_{i_1 \cdots i_n})$, where $0 < \lambda_j < 1$ for $j = 1, \ldots, p$ are the *ratio coefficients*.

These constructions are called *Moran constructions*. Moran discovered the formula $s = \dim_H F$, where s is the unique root of the equation

$$\sum_{i=1}^{p} \lambda_i^s = 1$$

Moran's major idea was to construct an optimal cover of the limit set (*Moran cover*) which is determined by the ratio coefficients. Our main insight into the Moran approach[27, 28] is that many of the strict conditions in the definition of a Moran construction are not required to build the Moran cover. For example, the geometric similarity of basic sets may be greatly weakened. Furthermore, although Moran only considered constructions modeled by the full shift, his approach can be generalized to constructions modeled by arbitrary symbolic dynamical systems. Our approach allows us to extend the original Moran idea to much broader classes of geometric constructions.

In particular, we introduce the *Moran-like constructions* defined as follows. Let (Q, σ) be a symbolic dynamical system where $Q \subset \Sigma_p^+$ is a compact shift invariant subset. Throughout the paper we assume that $\sigma \mid Q$ is topologically transitive. We allow basic sets $\Delta_{i_1 \cdots i_n}$ with n-tuples $i_1 \cdots i_n$ which are admissible with respect to Q such that:

(1) The following holds:

$$\underline{B}_{i_1 \cdots i_n} \subset \Delta_{i_1 \cdots i_n} \subset \bar{B}_{i_1 \cdots i_n}$$

where $\underline{B}_{i_1 \cdots i_n}$ and $\bar{B}_{i_1 \cdots i_n}$ are closed balls having radii $\underline{r}_{i_1 \cdots i_n}$ and $\bar{r}_{i_1 \cdots i_n}$, respectively.

(2) $\operatorname{int} \underline{B}_{i_1 \cdots i_n} \cap \operatorname{int} \underline{B}_{i'_1 \cdots i'_n} = \varnothing$ if $(i_1, ..., i_n) \neq (i'_1, ..., i'_n)$.

(3) $\underline{r}_{i_1 \cdots i_n} = C_1 \prod_{j=1}^n \lambda_{i_j}$ and $\bar{r}_{i_1 \cdots i_n} = C_2 \prod_{j=1}^n \lambda_{i_j}$, where $0 < \lambda_i < 1$, $i = 1, ..., p$, and C_1, C_2 are positive constants.

We stress that the topology and geometry of basic sets may be quite complicated. For example, they may not be connected and their boundary may be fractal. In particular the basic sets at level n of the construction need not be geometrically similar to the basic sets at level $n-1$. *Furthermore, the basic sets at a given level may intersect.* This class of constructions includes Moran geometric constructions. One very particular case is when a geometric construction is effected by a finite collection of similarity maps (affine contractions) $h_1, ..., h_p$ such that

$$\Delta_{i_1 \cdots i_n} = h_{i_1} \circ \cdots \circ h_{i_n}(\Delta)$$

where Δ denotes a ball in $\mathbb{R}^n$ (Fig. 3).

Given $\omega \in Q$, the intersection $\bigcap_{n=1}^{\infty} \Delta_{i_1 \cdots i_n}$ consists of a single point x. This produces a map $\chi: Q \to F$ defined by $\chi(\omega) = x$. It is a Hölder continuous map from Q onto F. To see this, let $\omega_1 = (i_1 i_2 \cdots i_n j \cdots)$ and $\omega_2 = (i_1 i_2 \cdots i_n k \cdots)$, $j \neq k$, two points in Q. We have

$$\rho(\chi(\omega_1), \chi(\omega_2)) \leqslant \prod_{j=1}^{n} \lambda_{i_j} \leqslant \lambda_{\max}^n \leqslant C \rho(\omega_1, \omega_2)^{\alpha}$$

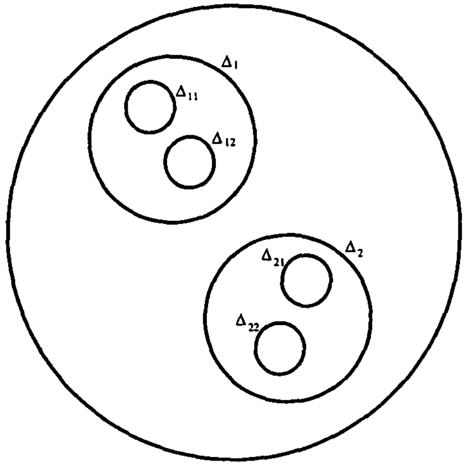

Fig. 3. Moran geometric construction with disjoint basic sets.

where $C > 0$ and $0 < \alpha < 1$ are constants and $\rho(\cdot, \cdot)$ denotes the Euclidean metric on F. Therefore, any Hölder continuous function on F pulls back to a Hölder continuous function on Q.

We assume that basic sets of a Moran-like geometric construction satisfy the following *separation condition*: for any two distinct n-tuples $(i_1 \cdots i_n)$ and $(j_1 \cdots j_n)$ we have

$$\Delta_{i_1 \cdots i_n} \cap \Delta_{j_1 \cdots j_n} \cap F = \varnothing$$

This separation condition allows significant overlaps of the basic sets on the same level. If this condition holds, the coding map χ is a homeomorphism and we can consider the induced map $G = \chi \circ \sigma \circ \chi^{-1}$ on the limit set F.

Since we consider geometric constructions which are modeled by a subshift of finite type (Σ_A^+, σ), the induced map G on the limit set is a local homeomorphism. Moreover, if one builds a geometric construction modeled by an arbitrary symbolic system (Q, σ) with the induced map on the limit set being expanding, then $\sigma \mid Q$ must be a subshift of finite type, i.e., $Q = \Sigma_A^+$ for some transfer matrix A. This follows from a result of Parry.[22]

The following theorem effects a complete multifractal analysis of Gibbs measures supported on limit sets of Moran-like geometric constructions modeled by subshifts of finite type. It is an immediate corollary of Theorem 4.

Theorem 5. Let G be the induced map on the limit set F for a Moran-like geometric construction modeled by a transitive subshift of finite type. Assume that the separation condition holds. Assume also that G is expanding and conformal [see (19) and (20)]. Then Statements 1–5 of Theorem 4 hold.

Consider a conformal self-similar geometric construction. Recall that this means that the basic sets $\Delta_{i_1 \cdots i_n}$ are given by

$$\Delta_{i_1 \cdots i_n} = h_{i_1} \circ \cdots \circ h_{i_n}(D)$$

where $h_1,..., h_p\colon D \to D$ are affine maps, i.e., $\rho(h_i(x), h_i(y)) = \lambda_i \rho(x, y)$ with $0 < \lambda_i < 1$ and $x, y \in D$ (a simply connected compact subset of $\mathbb{R}^m$). Assuming that basic sets Δ_i, $i = 1,..., p$, are disjoint, one can easily see that the induced map G on the limit set F is expanding and conformal [with $a(x) = \lambda_{i_1}^{-1}$ where $\chi(x) = (i_1 i_2 \cdots)$]. Thus, Theorem 4 applies. For some particular classes of measures (Bernoulli measures, self-similar measures, etc.) this result was obtained by several authors (see, for example, refs. 2, 7, 8, 20, 21, and 32).

Remarks. (1) A geometric construction satisfies the *open set condition* if and only if for all $n \in \mathbb{N}$, the interiors of all basic sets at level n are disjoint. We consider conformal selfsimilar geometric constructions satisfying the open set condition and where the basic sets at the first step $\Delta_i = h_i(D)$ satisfy $h_i(D) \subset \text{interior}(D)$. The following simple argument shows that Theorem 5 applies for these constructions.

Let $\mathscr{B} = \{x \in F \mid \#(\chi^{-1}(x)) \geq 2\}$, i.e., the set of points which do not have unique coding under χ. Since the construction is given by self-similar maps, we have

$$\begin{aligned}\mathscr{B} &\subset \bigcup_{n=1}^{\infty} \bigcup_{i_1 \cdots i_n} h_{i_1} \circ \cdots \circ h_{i_n}(\text{boundary}\,(D)) \\ &= \bigcup_{n=1}^{\infty} \bigcup_{i_1 \cdots i_n} (\text{boundary}\,(\Delta_{i_1 \cdots i_n}))\end{aligned}$$

However, our hypothesis that $h_i(D) \subset \text{interior}(D)$ implies that

$$F \cap \mathscr{B} \subset F \cap \bigcup_{n=1}^{\infty} \bigcup_{i_1 \cdots i_n} h_{i_1} \circ \cdots \circ h_{i_n}(\text{boundary}\,(D)) = \varnothing$$

This shows that all points.in the limit set F have unique codings and thus the induced map G is defined on all of F. Hence Theorem 4 applies.

Lau and Ngai[17] effected a type of multifractal analysis of self-similar measures on the limit sets of self-similar constructions where some type of overlaping of basic sets is allowed. They only work with upper Rényi dimension and thus avoid the nontrivial issue of the limit (defining the Rényi dimension) existing.

(2) Consider again a self-similar geometric construction modeled by the full shift σ and assume that ν is the Bernoulli measure defined by the vector $(p_1,..., p_r)$, where $0 < p_k < 1$ and $\sum_{k=1}^r p_k = 1$. It follows from results in ref. 27 that

$$P_{\Sigma_p^+}(\log(\lambda_{i_1}^{T(q)} p_{i_1}^q)) = 0$$

is equivalent to

$$\sum_{k=1}^{r} \lambda_i^{T(q)} p_i^q = 1$$

An easy calculation shows directly that $T(0) = d$, $T(1) = 0$, and that $T(q)$ is either linear or strictly convex.

(3) One can obtain a more general class of geometric constructions (more general than self-similar geometric constructions) to which Theorem 4 can be applied by considering a geometric construction effected by p sequences of bi-Lipschitz contraction maps $h_i^{(n)}: D \to D$ such that

$$\Delta_{i_1 \cdots i_n} = h_{i_1}^{(1)} \circ h_{i_2}^{(2)} \circ \cdots \circ h_{i_n}^{(n)}(D)$$

and for any $x, y \in D$,

$$\underline{\lambda}_i^{(n)} \operatorname{dist}(x, y) \leqslant \operatorname{dist}(h_i^{(n)}(x), h_i^{(n)}(y)) \leqslant \bar{\lambda}_i^{(n)} \operatorname{dist}(x, y)$$

where $0 < \underline{\lambda}_i^{(n)} \leqslant \bar{\lambda}_i^{(n)} < 1$ are Lipschitz constants for the maps $(h_i^{(n)})^{-1}$ and $h_i^{(n)}$, respectively.[28] We assume that the Lipschitz constants admit the following *asymptotic estimates*: there exist $0 < \lambda_i < 1$ such that

$$\left| \frac{\underline{\lambda}_i^{(n)}}{\lambda_i} - 1 \right| \leqslant e^{-n}, \qquad \left| \frac{\bar{\lambda}_i^{(n)}}{\lambda_i} - 1 \right| \leqslant e^{-n} \tag{21}$$

One can check that the induced map G is expanding and conformal [with $a(x) = \lambda_{i_1}^{-1}$, where $\chi(x) = (i_1 i_2 \cdots)$] and Theorem 4 applies.

Theorem 11 in ref. 27 shows there exists a geometric construction effected by two sequences of bi-Lipschitz contraction maps which do *not* satisfy the asymptotic estimates (21). Although the basic sets at each step of the construction are disjoint and the induced map G on the limit set F

is expanding, this geometric construction does *not* admit the multifractal analysis described by Theorem 5. Theorem 11 in ref. 27 provides two subsets A and B in F of positive measure for which the pointwise dimension exists for almost all points and takes on two distinct values α_1 and α_2. This implies that $f_\nu(\alpha_i) = \alpha_i$, $i = 1, 2$, which contradicts the fact that the equation $f_\nu(\alpha) = \alpha$ has the unique root $\alpha = \alpha(1)$. This construction yields a Hölder continuous homeomorphism of a compact metric space with an ergodic invariant measure of positive entropy for which the dimension spectrum is not convex, and hence the multifractal formalism fails.

It is still an open problem in dimension theory whether one can effect the complete multifractal analysis of Gibbs measures supported on the limit set of a Moran geometric construction modeled by a transitive subshift of finite type. By Theorem 9 in ref. 27, the pointwise dimension of such a measure exists almost everywhere.

(4) Theorem 1 is valid for any Moran geometric construction modeled by a symbolic system on which the pressure function is smooth. One of the reasons why we require the symbolic model (Q, σ) to be a subshift of finite type is that the smoothness of the pressure function is essentially known only in this case.

(5) We stress again that we have not used any techniques from the theory of large deviations to prove Theorem 1. However, Rolf Riedi explained to us that by combining our smoothness and convexity results for $T(q)$ with (5), we have verified all the hypotheses needed to apply a large-deviation theorem of Ellis and obtain an interesting formula for the dimension spectrum.

More precisely, consider the random variable $X_n = \log \nu(\Delta_r^n)$, where n has been picked uniformly from $1,..., N_r$. The moment generating function of X_n is

$$c_r(q) = \exp(qX_n) = (1/N_r) \sum_{\Delta_r^j \in \mathfrak{N}_r} \nu(\Delta_r^j)^q$$

Therefore, (5) implies that

$$\lim_{r \to 0} \frac{\log c_r(q)}{\log r} = T(q) - T(0)$$

which by Theorem 1 is smooth and convex. Thus, the assumptions of Theorem II.2 in ref. 6 are met with $a_n = \log q(1/r)$.

Recall that the Legendre transform of $T(q)$ is the (dimension spectrum) function $f_\nu(\alpha)$. The following theorem is a corollary of Ellis' theorem[32] and gives a *counting* approach to the multifractal analysis.

Theorem 6. Let ν be the equilibrium measure on F corresponding to a Hölder continuous function ξ. If $\nu \neq \nu_{\max}$ or $\nu_{\max} \neq m$, then

$$f_\nu(\alpha) = \lim_{\varepsilon \to 0} \lim_{r \to 0} \frac{\log N_r(\alpha, \varepsilon)}{\log(1/r)}$$

where $N_r(\alpha, \varepsilon)$ is the number of sets $\Delta_r^j \in \mathfrak{A}_r$ such that $\alpha - \varepsilon < \nu(\Delta_r^j) \leqslant \alpha + \varepsilon$.

Again we thank Rolf Riedi for supplying this remark.

(6) Consider the case when the measure m_λ is the measure of maximal entropy $m_\lambda = \nu_{\max}$. This implies that the function $a(x)$ is cohomological to a constant (in particular, $\lambda_{i_1} = \lambda = \text{const}$ for all i). If $\nu = m_\lambda = \nu_{\max}$, then the function ψ is cohomological to a constant as well. This implies that $T(q) = (1-q)\, d$. Since the pointwise dimension of m_λ is equal to s everywhere in F, we have that $f_\nu(d) = d$ and $f_\nu(\alpha) = 0$ for all $\alpha \neq d$.

APPENDIX A. THE DR PROPERTY OF EQUILIBRIUM MEASURES FOR CONTINUOUS EXPANDING MAPS

Let X be a compact metric space with metric ρ. We say that a continuous map $g: X \to X$ is *expanding* if g is a local homeomorphism and there exist constants $F \geqslant E > 1$ and $r_0 > 0$ such that

$$B(g(x), Er) \subset g(B(x, r)) \subset B(g(x), Fr) \tag{A1}$$

for every $x \in X$ and $0 < r < r_0$.

Without loss of generality we may assume that for any $x \in X$, the map g restricted to the ball $B(x, r_0)$ is a homeomorphism.

We recall that a *Markov partition* for an expanding map $g: X \to X$ is a finite cover of X by elements, called *rectangles*, $\{R_1, ..., R_p\}$, such that:

(1) Each rectangle R is the closure of its interior $\mathring{R}$.

(2) $\mathring{R}_i \cap \mathring{R}_j = \varnothing$ for $i \neq j$.

(3) Each $g(R_i)$ is a union of rectangles R_j.

We construct a special Markov partition for an expanding map such that the rectangle containing a given point in X is *almost* a ball. Let $R(x)$ denote the rectangle in $\mathscr{R}$ that contains the point x.

Theorem A1. There are positive constants C_1, C_2, and a positive integer k such that for any $0 < r \leqslant r_0$ and any $x \in X$ there exists a Markov partition $\mathscr{R}_x = \{R_1, ..., R_p\}$ for the map g^k such that $\operatorname{diam}(R_i) \leqslant C_2 r$ for all $i = 1, ..., M$ and $B(x, C_1 r) \subset R(x)$.

Proof. Let $k>1$ be an integer which we specify later. Fix a point $x \in X$ and choose r such that $l0r<(1/b^k)\,r_0$, where b is some positive number. Also choose a finite cover $\mathscr{B}^0$ of X by balls $B_i^0 = B(x_i, r)$ such that $x_1 = x$ and

$$\bigcup_{i>2} B_i^0 \cap B\left(x, \frac{3r}{4}\right) = \varnothing$$

Given i, consider a cover $\mathscr{C}_i^0$ of the set $g^k(B_i^0)$ by balls $B_j^0 \in \mathscr{B}^0$. Let

$$B_i^1 = \bigcup_{B_j^0 \in \mathscr{C}_i^0} g^{-k}(B_j^0)$$

Lemma A1. We have $B_i^1 \subset B(x_i, r+2a^{-k}r)$.

Proof of Lemma A1. Consider a ball $B_j^0 \in \mathscr{C}_i^0$ and a point $y \in B_j^0 \backslash g^k(B_i^0)$. Choose $z \in B_j^0 \cap g^k(B_i^0)$. Clearly the distance $\rho(z, y) \leqslant 2r$. By A1, $\rho(g^{-k}y, g^{-k}z) \leqslant a^{-k}2r$. The lemma follows since $g^{-k}z \in B_i^0$. ▌

Consider the cover $\mathscr{B}^1$ of X by sets $\{B_i^1\}$. Given i, we have

$$g^k(B_i^1) = \bigcup_{B_j^0 \in \mathscr{C}_i^0} B_j^0 \tag{A2}$$

Let $\mathscr{C}_i^1$ be the cover of the set $g^k(B_i^1)$ by sets $B_j^1 \in \mathscr{B}^1$ with $B_j^0 \in \mathscr{C}_i^0$. Set

$$B_i^2 = \bigcup_{B_j^1 \in \mathscr{C}_j^1} g^{-k}(B_j^1)$$

Lemma A2. We have $B_i^2 \subset B(x_i, r+2a^{-k}r+2a^{-2k}r)$.

Proof of Lemma A2. Consider a set $B_j^1 \in \mathscr{C}_i^1$ and a point $y \in B_j^1 \backslash g^k(B_i^1)$. Choose $z \in B_j^1 \cap g^k(B_i^1)$. Clearly the distance $\rho(z, y) \leqslant 2r$. By (A2) and Lemma A1, we have $\rho(y, z) \leqslant 2A^{-k}r$. The lemma follows. ▌

By induction we construct covers $\mathscr{B}^n = \{B_i^n\}$, $n = 2, 3, \ldots$, with the following properties:

(1) $g^k(B_i^n) = \bigcup_{B_j^0 \in \mathscr{C}_i^0} B_j^{n-1}$.

(2) $B_i^n \subset B(x_i, r + 2r\sum_{l=1}^n A^{-lk})$.

We consider the cover $\mathscr{B}^\infty$ which consists of the sets

$$B_i^\infty = \bigcup_{n=0}^{\infty} B_i^n$$

Lemma A3. We have that:

(1) $g^k(B_i^\infty) = \bigcup_{B_j'' \in \mathscr{C}_i''} B_j^\infty$.

(2) $B_i^\infty \subset B(x_i, r(1 + 2A^{-k}/(1 - A^{-k})))$.

(3) For sufficiently large k the set $B_i^\infty \subset B(x_i, (5/4)\,r)$.

(4) $\bigcup_{i \geqslant 2} B_i^\infty \cap B(x, r/4) = \varnothing$.

Proof of Lemma A3. This first two statements are an immediate consequence of the above properties 1 and 2 of covers $\mathscr{B}''$. In. Statements 3 and 4 follow directly from them. ∎

The first statement of Lemma A3 means the cover $\mathscr{B}^\infty$ is a *Markov cover*, i.e., its elements satisfy properties 1 and 3 in the definition of Markov partition. We will cut elements of this Markov cover to obtain the desired Markov partition.

Given $y \in X$, let $s(y) = (i_1 \cdots i_n)$ be the set of integers such that $y \in B_{i_j}^\infty$. Set

$$R(y) = \bigcap_{i_j \in s(y)} B_{i_j}^\infty$$

Lemma A4. (1) For every $y \in X$, the set $R(y)$ is open.

(2) If $z \in R(y)$, then $R(z) \subset R(y)$.

(3) If $z \notin R(y)$, then $R(z) \cap R(y) = \varnothing$.

(4) For every $z \in X$, we have $R(g^k(z)) \subset g^k(R(z))$.

Proof of Lemma A4. The first statement is obvious since the sets B_i^∞ are open. Now assume that $z \in R(y)$. Then $z \in B_{i_j}^\infty$ for every $i_j \in s(y)$ and $s(y) \subset s(z)$. Hence

$$R(z) = \bigcap_{i_j \in s(z)} B_{i_j}^\infty \subset \bigcap_{i_j \in s(y)} B_{i_j}^\infty$$

Now assume that $z \notin R(y)$. If there exists $w \in R(z) \cap R(y)$, then by Statement 2 we have $R(z) \subset R(y)$. Thus $z \in R(y)$ and we obtain a contradiction.

To prove the last statement, consider a point $z \in R(z) = \bigcap_{i_j \in s(z)} B_{i_j}^\infty$. Then $g^k(z) \in \bigcap_{i_j \in s(z)} g^k(B_{i_j}^\infty)$. By Statement 1 of Lemma 3, $g^k(z) \in \bigcap_{i_l \in s(g^k(z))} B_{i_l}^\infty$ and hence

$$R(g^k(z)) \subset \bigcap_{i_l \in s(g^k(z))} B_{i_l}^\infty \subset \bigcap_{i_j \in s(z)} g^k(B_{i_j}^\infty)$$

This completes the proof of Lemma A4. ∎

Lemma A4 implies that there exists a cover $\mathscr{R}_x$ of X by closed sets $\{R_1, ..., R_p\}$ and an integer $1 \leqslant k \leqslant M$ satisfying the following:

(1) For every $1 \leqslant j \leqslant k$ and every $z \in \mathring{R}_j$ we have $R_j = \overline{R(z)}$.

(2) For every $k+1 \leqslant j \leqslant N$ there exist finitely many points $y_{jl} \in X$ such that for every $z \in \mathring{R}_j$, we have $R_j = \overline{R(z)} \setminus \bigcup_l R(y_{jl})$.

We claim that the cover $\mathscr{R}_x$ is a Markov partition for g^k. We need only check Property 3 in the definition of Markov partition, since the other properties follow from Lemmas A3 and A4.

Given a set R_i and a point $z \in \mathring{R}_i$, assume that $g^k(z) \in \mathring{R}_j$.

If $1 \leqslant i \leqslant k$, then $R_i = R(z)$. By Statement 4 of Lemma A4 we have that $R(g^k(z)) \subset g^k(R_i)$. Since $R_j \subset R(g^k(z)$, this implies the Markov property.

If $k+1 \leqslant i \leqslant N$, then $R_i = \overline{R(z)} \setminus \bigcup_l R(y_{il})$. By Statement 4 of Lemma A4 we have that $R(g^k(z)) \subset g^k(R(z))$. Since $R_\subset R(g^k(z))$, this implies that $R_j \subset g^k(R(z))$. Applying an appropriate branch of the inverse map g^{-k}, we have that $g^{-k}R_j \subset R(z)$. Assume that there is a point $w \in g^{-k}r_j$ which does not belong to R_i. Then $w \in R(y_{il})$ for some $y_{il} \in X$. This implies that $g^k(w) \in R_j$ and hence $R_j \subset R(g^k(w))$. By Statement 4 of Lemma A4, we have

$$g^{-k}(R_j) \subset g^{-k}(R(g^k(w))) \subset R(w) \subset R(y_{il})$$

This is impossible since $z \in g^{-k}(R_j)$ and the Markov property has been verified.

It follows directly from Statement 4 of Lemma A3 that the Markov partition $\mathscr{R}_x$ has the desired property with respect to the given point x. ▮

We use the special Markov partition constructed in Theorem A1 to prove the following statement.

Theorem A2. Let ϕ be a Hölder continuous function on X. Assume that g is conformal (see (20)). Then any equilibrium measure for ϕ with respect to g is diametrically regular [see condition (DR)].

Proof. Let μ_ϕ be an equilibrium measure for ϕ. Let also $\mathscr{R}$ be a Markov partition of X. Given $x \in X$ and a number $0 < r < r_0$, consider a Moran cover $\mathfrak{U}_r$ of X and choose those elements $R^{(1)}, ..., R^{(m)}$ from this cover that intersect the ball $B(x, 2r)$. We have that

(1) $R^{(j)} = R_{i_1 \cdots i_{n(x_j)}}$ $j = 1, ..., m$ where $x_j \in X$ is a point;

(2) $\operatorname{diam} R^{(j)} \leqslant r$, $j = 1, ..., m$;

(3) $m < K$ where K is a constant independent of x and r;

There is an element $R^{(i)}$ in the Moran cover that contains x. We have that

$$R^{(i)} \subset B(x, r) \subset B(x, 2r) \subset \bigcup_{j=1}^{m} R^{(j)} \tag{A3}$$

Define the function ψ such that $\log \psi = \phi - P(\phi)$. Clearly, ψ is a Hölder continuous function on X such that $P(\log \psi) = 0$ and μ_ϕ is the Gibbs measure for $\log \psi$. We have for any $j = 1,..., m$, that [see Appendix B, Eq. (B1)]

$$C_2 \prod_{k=0}^{n(x_j)-1} \psi(g^k(x_j)) \leqslant \mu_\phi(R^{(j)}) \leqslant C_1 \prod_{k=0}^{n(x_j)-1} \psi(g^k(x_j)) \tag{A4}$$

where $C_1 > 0$ and $C_2 > 0$ are constants. Since g is expanding, we also have that

$$C_4 \leqslant \frac{\prod_{k=0}^{n(x_i)-1} \psi(g^k(x_j))}{\prod_{k=0}^{n(x_i)-1} \psi(g^k(x))} \leqslant C_5 \tag{A5}$$

where $C_3 > 0$ and $C_4 > 0$ are constants. It now follows from (A3)–(A5) that

$$\mu_\phi(B(x, 2r)) \leqslant \mu_\phi\left(\bigcup_{j=1}^{m} R^{(j)}\right) \leqslant K \prod_{k=0}^{n(x_i)-1} \psi(g^k(x))$$

$$\leqslant C_5 \mu_\phi(R^{(j)}) \leqslant C_5 \mu_\phi(B(x, r))$$

where $C_5 > 0$ is a constant. This completes the proof. ∎

APPENDIX B. FACTS ABOUT PRESSURE

This Appendix contains some essential definitions and facts from symbolic dynamics and thermodynamic formalism. For details consult refs. 1 and 34. Let X denote a compact metric space and let $C(X)$ denote the space of real-valued continuous functions on X.

(1) Let $g: X \to X$ be a continuous map. We define the *pressure* P: $C(X) \to \mathbb{R}$ defined by

$$P(\phi) = \sup_{\mu \in \mathfrak{M}(X)} \left(h_\mu(g) + \int_X \phi \, d\mu \right)$$

where $\mathfrak{M}(X)$ denotes the set of shift-invariant probability measures on X and $h_\mu(f)$ denotes the Kolmogorov–Sinai entropy of the map g with respect to the measure μ. A Borel probability measure $\mu = \mu_\phi$ on X is called an *equilibrium measure* for the potential $\phi \in C(X)$ if

$$P(\phi) = h_\mu(g) + \int_X \phi \, d\mu$$

(2) The pressure function $P: C^\alpha(\Sigma_A^+, \mathbb{R}) \to \mathbb{R}$ is real analytic. We remark that this result may not be true if Σ_A^+ is replaced by an arbitrary symbolic system.

(3) Let $\phi \in C^\alpha(\Sigma_A^+, \mathbb{R})$. The map $\mathbb{R} \to \mathbb{R}$ defined by $t \to P(t\phi)$ is convex. It is strictly convex unless ϕ is cohomologous to a constant, i.e., there exist $C > 0$ and $g \in C^\alpha(\Sigma_A^+, \mathbb{R})$ such that $\phi(x) = g(\sigma x) - g(x) + C$.

(4) Let $\phi \in C(\Sigma_A^+)$. A Borel probability measure $\mu = \mu_\phi$ on Σ_A^+ is called a *Gibbs measure* for the potential ϕ if there exist constants $D_1, D_2 > 0$ such that

$$D_1 \leqslant \frac{\mu\{y: y_i = x_i,\ i = 0, \dots, n-1\}}{\exp(-nP(\phi) + \sum_{k=0}^{n-1} \phi(\sigma^k x))} \leqslant D_2$$

for all $x = (x_1 x_2 \cdots) \in \Sigma_A^+$ and $n \geqslant 0$. For subshifts of finite type, Gibbs measures exist for any Hölder continuous potential ϕ, are unique, and coincide with the equilibrium measure for ϕ.

(5) Given two continuous functions h_1 and h_2 on Σ_A^+, we have

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} P(h_1 + \varepsilon h_2) = \int_{\Sigma_A^+} h_2 \, d\mu_{h_1} \tag{B1}$$

where μ_{h_1} denotes the Gibbs measure for the potential h_1.

APENDIX C. FACTS ABOUT THE LEGENDRE TRANSFORM

Let f be a C^2 strictly convex map on an interval I, hence, $f''(x) > 0$ for all $x \in I$. The *Legendre transform* of f is the function g of a new variable p defined by

$$g(p) = \max_{x \in I} (px - f(x))$$

It is easy to show that g is strictly convex and that the Legendre transform is involutive. One can also show that strictly convex functions f and g form a Legendre transform pair if and only if $g(\alpha) = f(q) + q\alpha$, where $\alpha(q) = -f'(q)$ and $q = g'(\alpha)$. See ref. 33 for more details.

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